Fundamentals of Financial Arythmetics

Lecture 2

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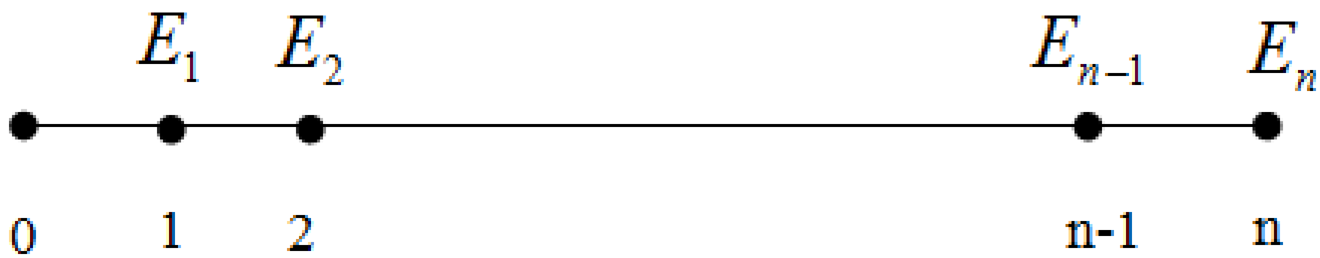
- An annuity is a series of equal payments at regular intervals (deposits to a savings account, home mortgage payments).
- Payment period – interval between annuity payments.
- Term – the fixed period of time for which payments are made.
- Payment periods and compounding periods.

- There are two types of annuities:
  - **Annuity-immediate** – payments are made at the end of payment periods (at the end of each month, quarter, semi-year, year etc.).
  - **Annuity due** – payments are made at the beginning of payment periods (at the beginning of each month, quarter, semi-year, year etc.)

- The aim of lecture is to the find future value of annuity.
- **Future value of annuity is a sum of the future values of all payments**.

- To find the future value of annuity we can use simple interest, compound interest or continuously compounded interest.

Annuity – simple interest



Annuity-immediate

$$K_n = E_1 + E_2 + \cdots + E_n + E_1(n-1)r + E_2(n-2)r + \cdots + E_{n-1}r$$

E_1, E_2 Payments at the end of the first period and at the end of the second period, respectively

r – interest rate adjusted to the period of payments (payments each month – monthly interest rate, payments each year – annual interest rate)

- First, we have to find the value of each payment at the end of a given period using simple interest (we can compare values only at the same time, for instance at the end of the last period, at the beginning of the first period and so on).We choose the end of n-th period.
- The future value of the first payment is

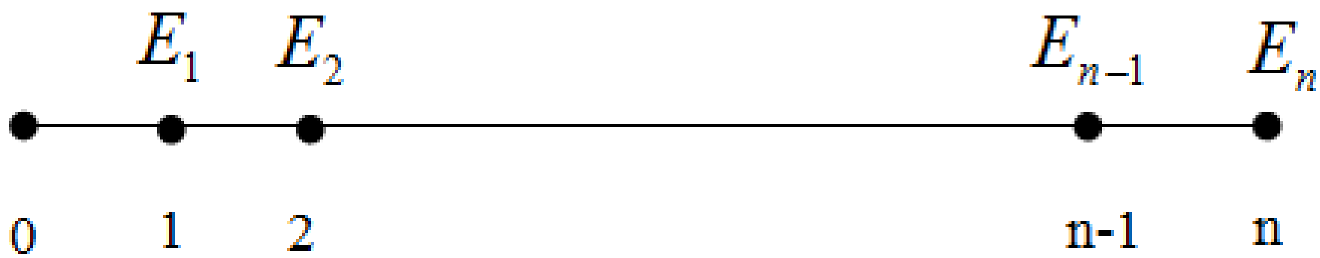
$$E_1 + (n-1) \cdot r \cdot E_1$$

- Comment: between the end of the first period and the end of the n-th period are n-1 periods (the end of January and the end of the end of December).
- The future value of the second payment is

$$E_2 + (n-2) \cdot r \cdot E_2$$

Finally we have to add all future values using a sum of arithmetic sequence.

Annuity – simple interest



Annuity due

$$K_n = E_1 + E_2 + \cdots + E_n + E_1 n \cdot r + E_2 (n-1) r + \cdots + E_n r$$

E_1, E_2 Payments at the beginning of the first period and at the beginning of the second period, respectively

- First we have to find the value of each payment at the end of n-th period.
- The future value of the first payment is

$$E_1 + n \cdot r \cdot E_1$$

- Comment: between the beginning of the first period and the end of the n-th period are n periods (the beginning of January and the end of the end of December).

- The future value of the second payment is

$$E_2 + (n-1) \cdot r \cdot E_2$$

- Finally we have to add all future values using a sum of arithmetic sequence.

Annuity – simple interest

level payment annuity **E**

Annuity-immediate | **Annuity due**

Future value

Annuity-immediate:

$$K_n = E \cdot n \cdot \left(1 + \frac{n-1}{2} r\right)$$

Annuity due:

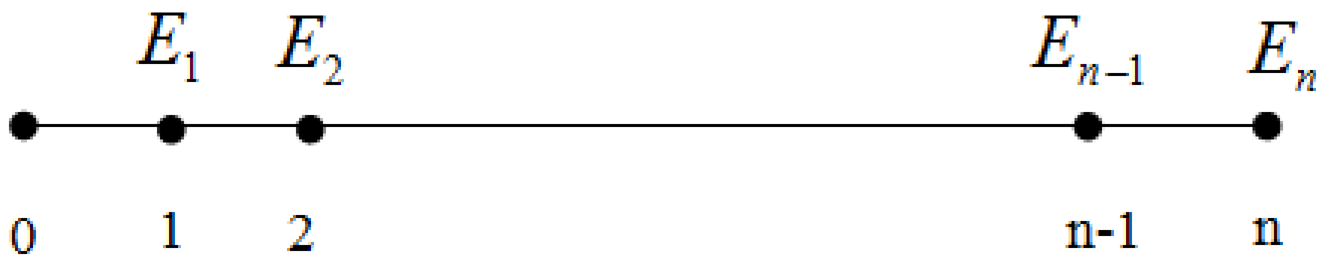
$$K_n = E \cdot n \cdot \left(1 + \frac{n+1}{2} r\right)$$

Present value

$$K_0 = E \cdot n \cdot \left(1 + \frac{n \pm 1}{2} r\right) \frac{1}{1 + n \cdot r}$$

$$K_t = E \cdot n \cdot \left(1 + \frac{n \pm 1}{2} r\right) \frac{1 + t \cdot r}{1 + n \cdot r} \qquad t \in (0, n)$$

Annuity – compound interest



Annuity-immediate

$$K_n = E_1(1+r)^{n-1} + E_2(1+r)^{n-2} + \cdots + E_{n-1}(1+r) + E_n$$

Annuity due

$$K_n = E_1(1+r)^{n} + E_2(1+r)^{n-1} + \cdots + E_{n}(1+r)$$

To add all future values we use a sum of geometric sequence

Annuity – compound interest

level payment annuity **E**

Annuity-immediate | **Annuity due**

Future value

$$K_n = E \cdot \frac{(1+r)^n - 1}{r}$$

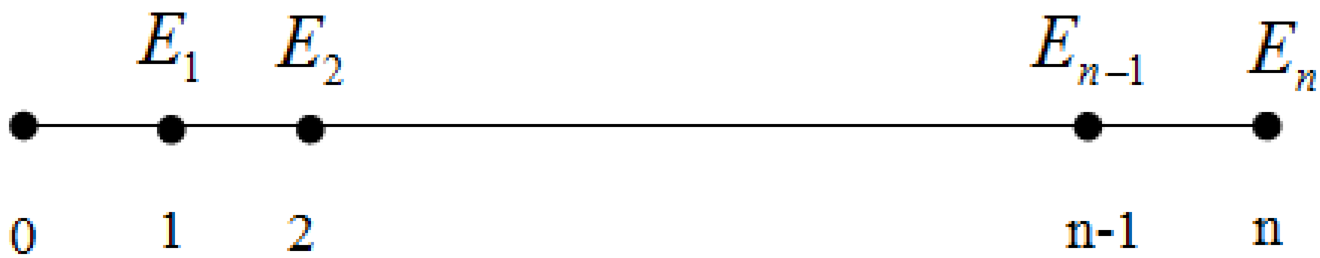
$$K_n = E \cdot (1+r)\frac{(1+r)^n - 1}{r}$$

Present value

$$K_0 = \frac{K_n}{(1+r)^n}$$

$$K_t = \frac{K_n}{(1+r)^{n-t}} \quad t \in (0, n)$$

Annuity – continuously compounded interest



Annuity-immediate

$$K_n = E_1 \cdot e^{(n-1)r} + E_2 \cdot e^{(n-2)r} + \cdots + E_{n-1} \cdot e^{r} + E_n$$

Annuity due

$$K_n = E_1 \cdot e^{n \cdot r} + E_2 \cdot e^{(n-1)r} + \cdots + E_n \cdot e^{r}$$

To add all future values we use a sum of geometric sequence

Annuity – continuously compounded interest

level payment annuity **E**

Annuity-immediate

$$K_n = E \cdot \frac{e^{n \cdot r} - 1}{e^r - 1}$$

$$K_0 = E \cdot \frac{1 - e^{-n \cdot r}}{e^r - 1}$$

Annuity due

$$K_n = E \cdot e^r \cdot \frac{e^{n \cdot r} - 1}{e^r - 1}$$

$$K_0 = E \cdot e^r \cdot \frac{1 - e^{-n \cdot r}}{e^r - 1}$$

$$K_t = K_n \cdot e^{-(n-t) \cdot r} \qquad t \in (0, n)$$

Example 1 – Annuity-immediate

- Calculate the present value (PV) of an annuity-immediate of amount 100 PLN paid monthly for a year at the rate of simple interest of 9% per annum. Also calculate its future value (FV) at time 1 year.

$$PV = E \cdot n \cdot \left(1 + \frac{n-1}{2} r\right) \frac{1}{1 + n \cdot r} \qquad FV = E \cdot n \cdot \left(1 + \frac{n-1}{2} r\right)$$

$$PV = 100 \cdot 12 \cdot \left(1 + \frac{12-1}{2} \cdot \frac{0.09}{12}\right) \frac{1}{1 + 0.09} = 1146.33$$

$$FV = 100 \cdot 12 \cdot \left(1 + \frac{12-1}{2} \cdot 0.0075\right) = 1249.5$$

Example 2a – Annuity-immediate – future value Compound interest

- Find the accumulated value of a 10-year annuity-immediate of 100 PLN per year if the effective rate of interest is 6% for the first 6 years and 4% for the last 4 years.

$$K_n = E \cdot \frac{(1+r)^n - 1}{r}$$

$$100 \cdot \frac{(1+0.06)^6 - 1}{0.06}(1.04)^4 + 100 \cdot \frac{(1+0.04)^4 - 1}{0.04} = 1240.66$$

Example 2b – Annuity-immediate – future value Compound interest

- Rework example above if the first 6 payments are invested at a rate of interest 6% and if the final 4 payments are invested at 4%.

$$K_n = E \cdot \frac{(1+r)^n - 1}{r}$$

$$100 \cdot \frac{(1+0.06)^6 - 1}{0.06}(1.06)^4 + 100 \cdot \frac{(1+0.04)^4 - 1}{0.04} = 1305.26$$

Example 3 – Annuity-immediate – present value

- Find the present value of an annuity which pays 500 PLN at end of each half-year for 10 years if the rate of interest is 9% convertible semiannually.

$$PV \equiv K_0 = E \cdot \frac{1-(1+r)^{-n}}{r}$$

$$PV = 500 \cdot \frac{1-(1+0.045)^{-20}}{0.045} = 6503.97$$

Example 4 – Annuity-immediate

- Determine the present value of receiving 1000 PLN per year for the next 7 years if it is continuously discounted at 6%.

$$PV = E \cdot \frac{1-e^{-n \cdot r}}{e^{r}-1}$$

$$PV = 1000 \cdot \frac{1-e^{-7 \cdot 0.06}}{e^{0.06}-1} = 5546.124$$

Example 5

- Find the accumulated value at end of 6 years of investment fund in which 100 PLN is deposited at the beginning of each month and 200 PLN is deposited at the end of each half-year, if the rate of interest is 12% convertible quarterly.

$$FV = E \cdot (1+r)\frac{(1+r)^n - 1}{r}$$

$$FV = E \cdot \frac{(1+r)^n - 1}{r}$$

Example 5

- Find the accumulated value at end of 6 years of investment fund in which 100 PLN is deposited at the beginning of each month and 200 PLN is deposited at the end of each half-year, if the rate of interest is 12% convertible quarterly.

$$r=\left(1+\frac{0.12}{4}\right)^{\frac{1}{3}}-1=0.0099 \qquad r=\left(1+\frac{0.12}{4}\right)^{2}-1=0.0609$$

$$FV=100\cdot 1.0099\cdot\frac{(1.0099)^{72}-1}{0.0099} \qquad FV=200\cdot\frac{(1.0609)^{12}-1}{0.0609}$$

$$10533.82+3391.77=13925.59$$

Example 6

- An investor wishes to accumulate 100 PLN. How many years must the investor pay 10 PLN at the beginning of each year if the interest rate is 12% per annum, compounding annually.
- Solve the problem of non-integer value of the term of annuities.

Example 6 – term of annuity due is not an integer

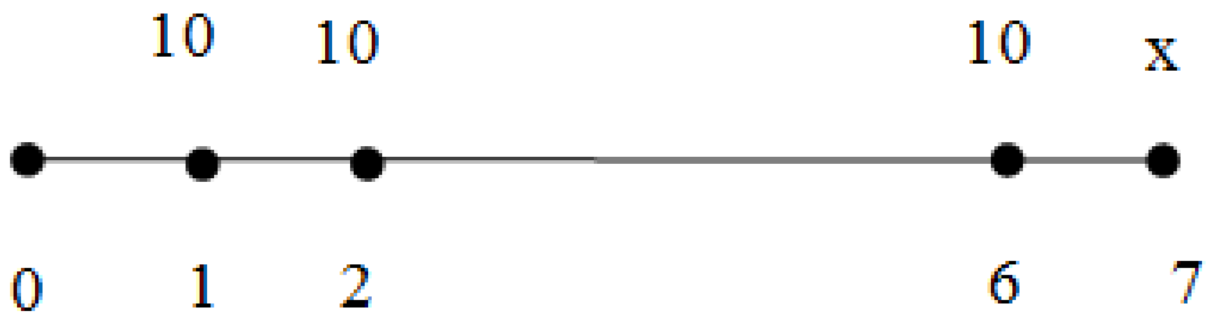
$E = 10$ $K_n = 100$ $r = 12\%$

$$K_n = E \cdot (1+r)\frac{(1+r)^n - 1}{r}$$

$$n = \frac{\ln\left(\frac{r \cdot K_n}{E \cdot (1+r)} + 1\right)}{\ln(1+r)} = \frac{\ln\left(\frac{0.12 \cdot 100}{10 \cdot 1.12} + 1\right)}{\ln(1.12)}$$

$$\boxed{n = 6.43}$$

Additional payment x



$$K_6 = 90.89 \qquad K_6 + x = 100$$

Equation of value at
the beginning of 7 year

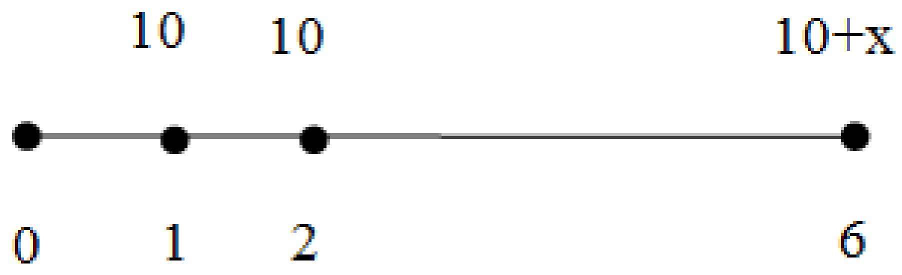
$$x = 9.11$$

$$(K_6 + x)\cdot(1 + r) = 100$$

Equation of value
at the end of 7 year

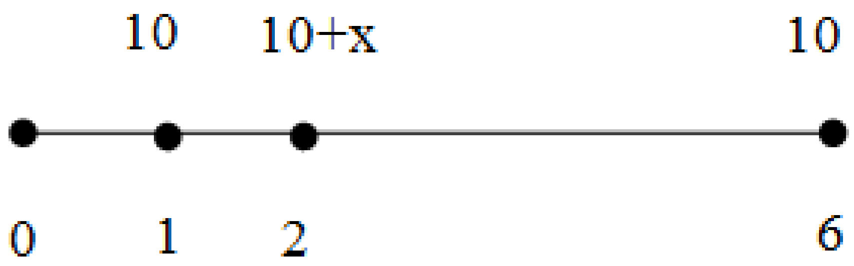
$$x = -1.60$$ reject

Enlargement of one of the payments



$$K_6 + x \cdot (1 + r) = 100$$

$$x = 8.13$$



$$K_6 + x \cdot (1 + r)^5 = 100$$

$$x = 5.17$$

New payments

$$E = \frac{r \cdot K_n}{(1+r)\left((1+r)^n - 1\right)}$$

$n = 6$ $\qquad E = 11.00$

Example 7

- An investor wishes to accumulate 100 PLN. How many years must the investor pay 10 PLN at the end of each year if the interest rate is 12% per annum, compounding annually.
- Solve the problem of non-integer value of the term of annuities.

Example 7 – term of annuity-immediate is not an integer

$E = 10$

$K_n = 100$

$r = 12\%$

$$K_n = E \cdot \frac{(1+r)^n - 1}{r}$$

$$\boxed{n = 6.96}$$

$$n = \frac{\ln\left(\frac{r \cdot K_n}{E} + 1\right)}{\ln(1+r)}$$

Additional payment x

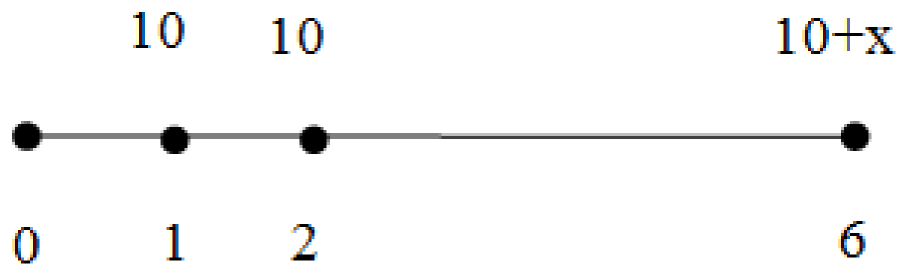
$$K_6 \cdot (1+r) + x = 100$$

$$K_n = E \cdot \frac{(1+r)^n - 1}{r}$$

$$x = 9.11$$

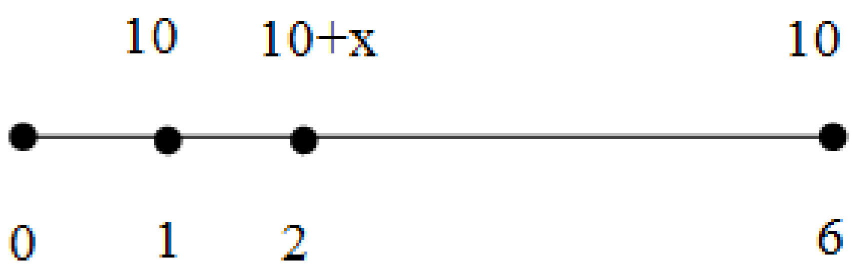
$$K_6 = 81.15$$

Enlargement of one of the payments



$$K_6 + x = 100$$

$$x = 18.85$$



$$K_6 + x \cdot (1+r)^4 = 100$$

$$x = 11.98$$

New payments

$$E = \frac{r \cdot K_n}{(1+r)^n - 1}$$

$n = 7$ $\qquad$ $E = 9.91$

Example 8

- An investor wishes to accumulate 100 PLN. How many years must the investor pay 10 PLN at the beginning of each year if the interest rate is 12% per annum (simple interest).
- Solve the problem of non-integer value of the term of annuities.

Example 8 – term of annuity due is not an integer, simple interest

$E = 10$ $K_n = 100$ $r = 12\%$

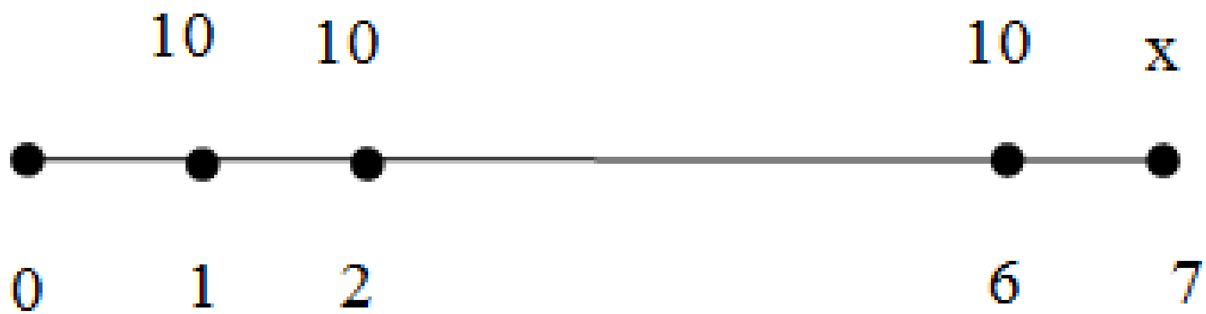
$$K_n = E \cdot n \cdot \left(1 + \frac{n+1}{2} r\right)$$

$$100 = 10 \cdot n \cdot \left(1 + \frac{n+1}{2} \cdot 0.12\right)$$

$$3 \cdot n^2 + 53 \cdot n - 500 = 0$$

$$\boxed{n = 6.81}$$

Additional payment x

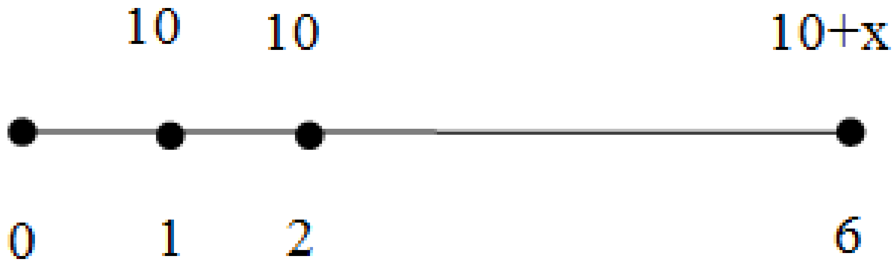


$$K_6 = 10 \cdot 6 \cdot \left(1 + \frac{6+1}{2} \cdot 0.12\right) = 85.2$$

$$K_6 + x = 100 \qquad x = 14.8$$

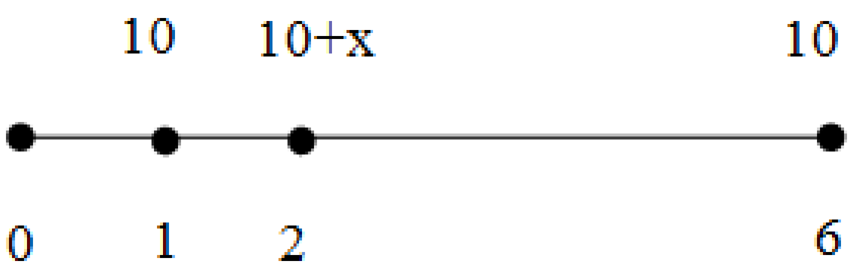
$$\left(K_6 + x\right) \cdot \left(1 + r\right) = 100 \qquad x = 4.09$$

Enlargement of one of the payments



$$K_6 + x \cdot (1 + r) = 100$$

$$x = 13.21$$



$$K_6 + x \cdot (1 + 5r) = 100$$

$$x = 9.25$$

$$K_6 = 85.2$$

New payments

$$E = \frac{K_n}{n \cdot \left(1 + \frac{n+1}{2} r\right)}$$

$$n = 7 \qquad E = 9.65$$