Fundamentals of Financial Arithmetic

Lecture 3

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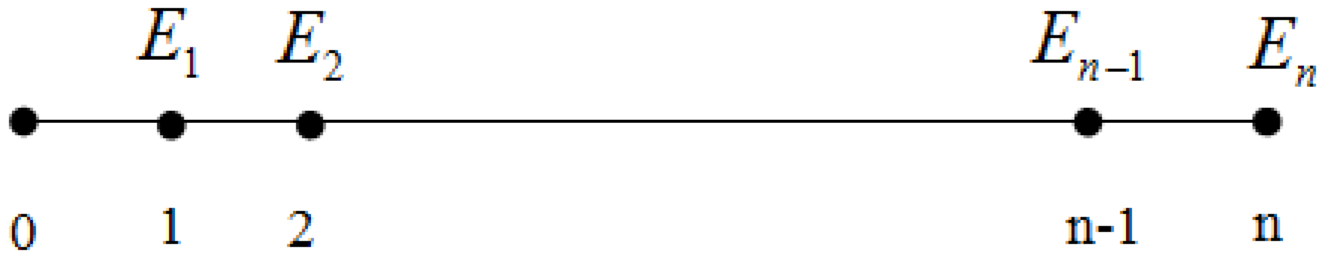
- An annuity is a series of equal payments at regular intervals (deposits to a savings account, home mortgage payments).
- Payment period – interval between annuity payments.
- Term – the fixed period of time for which payments are made.
- Payment periods and compounding periods.

- **Annuity-immediate** – payments are made at the end of payments period.
- **Annuity due** – payments are made at the beginning of payment periods.

Annuity – simple interest



Annuity-immediate

$$K_n = E_1 + E_2 + \cdots + E_n + E_1(n-1)r + E_2(n-2)r + \cdots + E_{n-1}r$$

Annuity due

$$K_n = E_1 + E_2 + \cdots + E_n + E_1 n \cdot r + E_2(n-1)r + \cdots + E_n r$$

Annuity – simple interest

level payment annuity **E**

Annuity-immediate **Annuity due**

Future value

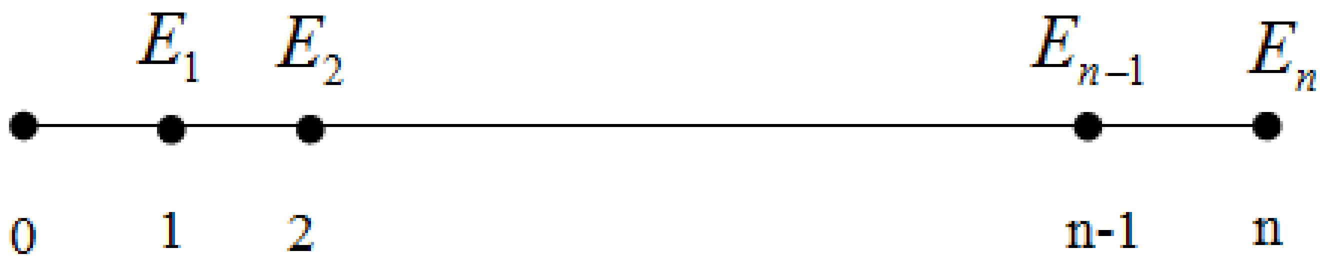
$$K_n = E \cdot n \cdot \left(1 + \frac{n-1}{2} r\right) \qquad K_n = E \cdot n \cdot \left(1 + \frac{n+1}{2} r\right)$$

Present value

$$K_0 = E \cdot n \cdot \left(1 + \frac{n \pm 1}{2} r\right) \frac{1}{1 + n \cdot r}$$

$$K_t = E \cdot n \cdot \left(1 + \frac{n \pm 1}{2} r\right) \frac{1 + t \cdot r}{1 + n \cdot r} \qquad t \in (0, n)$$

Annuity – compound interest



Annuity-immediate

$$K_n = E_1(1+r)^{n-1} + E_2(1+r)^{n-2} + \cdots + E_{n-1}(1+r) + E_n$$

Annuity due

$$K_n = E_1(1+r)^{n} + E_2(1+r)^{n-1} + \cdots + E_{n}(1+r)$$

Annuity – compound interest

level payment annuity **E**

Annuity-immediate **Annuity due**

Future value

$$K_n = E \cdot \frac{(1+r)^n - 1}{r}$$

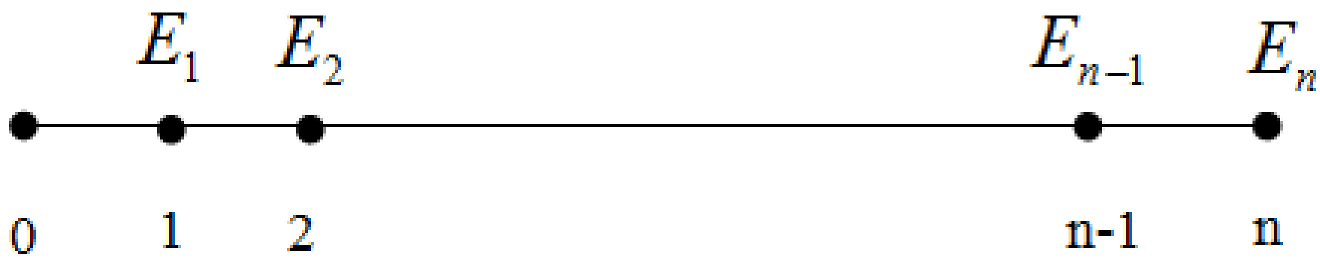
$$K_n = E \cdot (1+r)\frac{(1+r)^n - 1}{r}$$

Present value

$$K_0 = \frac{K_n}{(1+r)^n}$$

$$K_t = \frac{K_n}{(1+r)^{n-t}} \quad t \in (0, n)$$

Annuity – continuously compounded interest



Annuity-immediate

$$K_n = E_1 \cdot e^{(n-1)r} + E_2 \cdot e^{(n-2)r} + \cdots + E_{n-1} \cdot e^{r} + E_n$$

Annuity due

$$K_n = E_1 \cdot e^{n \cdot r} + E_2 \cdot e^{(n-1)r} + \cdots + E_n \cdot e^{r}$$

Annuity – continuously compounded interest

level payment annuity **E**

Annuity-immediate

Annuity due

$$K_n = E \cdot \frac{e^{n \cdot r} - 1}{e^r - 1}$$

$$K_n = E \cdot e^r \cdot \frac{e^{n \cdot r} - 1}{e^r - 1}$$

$$K_t = K_n \cdot e^{-(n-t) \cdot r} \qquad t \in (0, n)$$

$$K_0 = E \cdot \frac{1 - e^{-n \cdot r}}{e^r - 1}$$

$$K_0 = E \cdot e^r \cdot \frac{1 - e^{-n \cdot r}}{e^r - 1}$$

Example 1 – Annuity-immediate

- Calculate the present value of an annuity-immediate of amount 100 PLN paid monthly for a year at the rate of simple interest of 9% per annum. Also calculate its future value at time 1 year.

$$PV = E \cdot n \cdot \left(1 + \frac{n-1}{2} r\right) \frac{1}{1 + n \cdot r}$$

$$FV = E \cdot n \cdot \left(1 + \frac{n-1}{2} r\right)$$

Example 1– Annuity-immediate

- Calculate the present value of an annuity-immediate of amount 100 PLN paid monthly for a year at the rate of simple interest of 9% per annum. Also calculate its future value at time 1 year.

$$PV = E \cdot n \cdot \left(1 + \frac{n-1}{2} r\right) \frac{1}{1 + n \cdot r} \qquad FV = E \cdot n \cdot \left(1 + \frac{n-1}{2} r\right)$$

$$PV = 100 \cdot 12 \cdot \left(1 + \frac{12-1}{2} \cdot \frac{0.09}{12}\right) \frac{1}{1 + 0.09} = 1146.33$$

$$FV = 100 \cdot 12 \cdot \left(1 + \frac{12-1}{2} \cdot 0.0075\right) = 1249.5$$

Example 2a – Annuity-immediate – future value Compound interest

- Find the accumulated value of a 10-year annuity-immediate of 100 PLN per year if the effective rate of interest is 6% for the first 6 years and 4% for the last 4 years.

$$K_n = E \cdot \frac{(1+r)^n - 1}{r}$$

$$100 \cdot \frac{(1+0.06)^6 - 1}{0.06}(1.04)^4 + 100 \cdot \frac{(1+0.04)^4 - 1}{0.04} = 1240.66$$

Example 2b – Annuity-immediate – future value Compound interest

- Rework example above if the first 6 payments are invested at an effective rate of interest 6% and if the final 4 payments are invested at 4%.

$$K_n = E \cdot \frac{(1+r)^n - 1}{r}$$

$$100 \cdot \frac{(1+0.06)^6 - 1}{0.06}(1.06)^4 + 100 \cdot \frac{(1+0.04)^4 - 1}{0.04} = 1305.26$$

Example 3 – Annuity-immediate – present value

- Find the present value of an annuity which pays 500 PLN at end of each half-year for 10 years if the rate of interest is 9% convertible semiannually.

$$PV \equiv K_0 = E \cdot \frac{1-(1+r)^{-n}}{r}$$

$$PV = 500 \cdot \frac{1-(1+0.045)^{-20}}{0.045} = 6503.97$$

Example 4 – Annuity-immediate

- Determine the present value of receiving 1000 PLN per year for the next 7 years if it is continuously discounted at 6%.

$$PV = E \cdot \frac{1-e^{-n \cdot r}}{e^{r}-1}$$

$$PV = 1000 \cdot \frac{1-e^{-7 \cdot 0.06}}{e^{0.06}-1} = 5546.124$$

Example 5

- Find the accumulated value at end of 6 years of investment fund in which 100 PLN is deposited at the beginning of each month and 200 PLN is deposited at the end of each half-year, if the rate of interest is 12% convertible quarterly.

$$FV = E \cdot (1+r)\frac{(1+r)^n - 1}{r}$$

$$FV = E \cdot \frac{(1+r)^n - 1}{r}$$

Example 5

- Find the accumulated value at end of 6 years of investment fund in which 100 PLN is deposited at the beginning of each month and 200 PLN is deposited at the end of each half-year, if the rate of interest is 12% convertible quarterly.

$$r = \left(1 + \frac{0.12}{4}\right)^{\frac{1}{3}} - 1 = 0.0099 \qquad r = \left(1 + \frac{0.12}{4}\right)^{2} - 1 = 0.0609$$

$$FV = 100 \cdot 1.0099 \cdot \frac{(1.0099)^{72} - 1}{0.0099} \qquad FV = 200 \cdot \frac{(1.0609)^{12} - 1}{0.0609}$$

$$10533.82 + 3391.77 = 13925.59$$

Example 6a – term of annuity due is not an integer

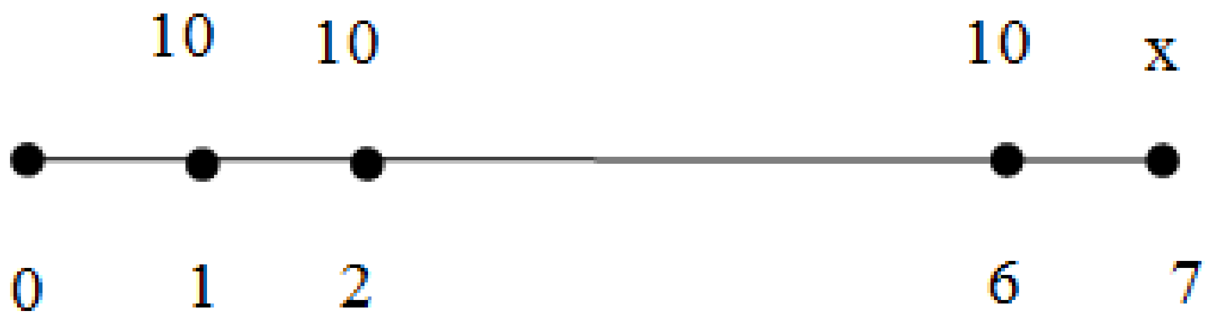
$E = 10$ $K_n = 100$ $r = 12\%$

$$K_n = E \cdot (1+r)\frac{(1+r)^n - 1}{r}$$

$$n = \frac{\ln\left(\frac{r \cdot K_n}{E \cdot (1+r)} + 1\right)}{\ln(1+r)} = \frac{\ln\left(\frac{0.12 \cdot 100}{10 \cdot 1.12} + 1\right)}{\ln(1.12)}$$

$$n = 6.43$$

Additional payment



$$K_6 = 90.89 \qquad K_6 + x = 100$$

Equation of value at the beginning of 7 year

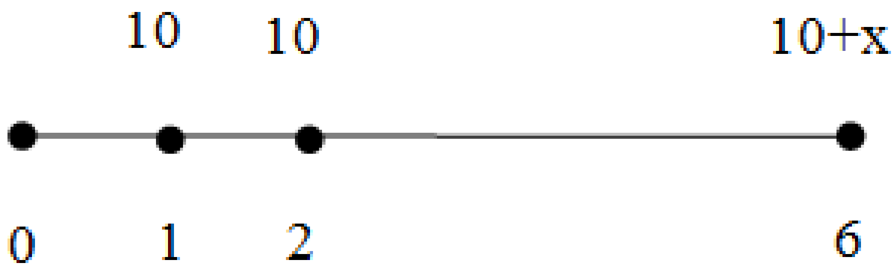
$$x = 9.11$$

$$\left(K_6 + x\right)\cdot\left(1 + r\right) = 100$$

Equation of value at the end of 7 year

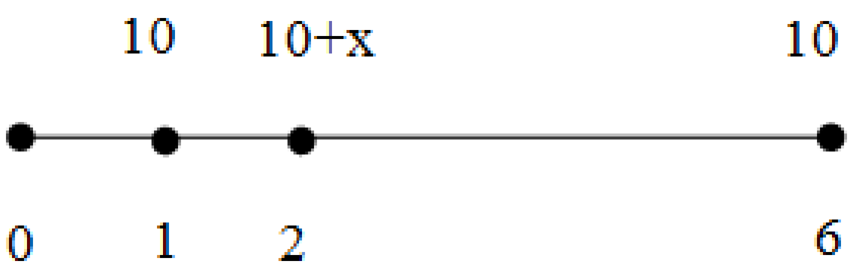
$$x = -1.60$$

Enlargement of one of the payment



$$K_6 + x \cdot (1+r) = 100$$

$$x = 8.13$$



$$K_6 + x \cdot (1+r)^5 = 100$$

$$x = 5.17$$

New payments

$$E = \frac{r \cdot K_n}{(1+r)\left((1+r)^n - 1\right)}$$

$n = 6$ $E = 11.00$

Example 6b – term of annuity-immediate is not an integer

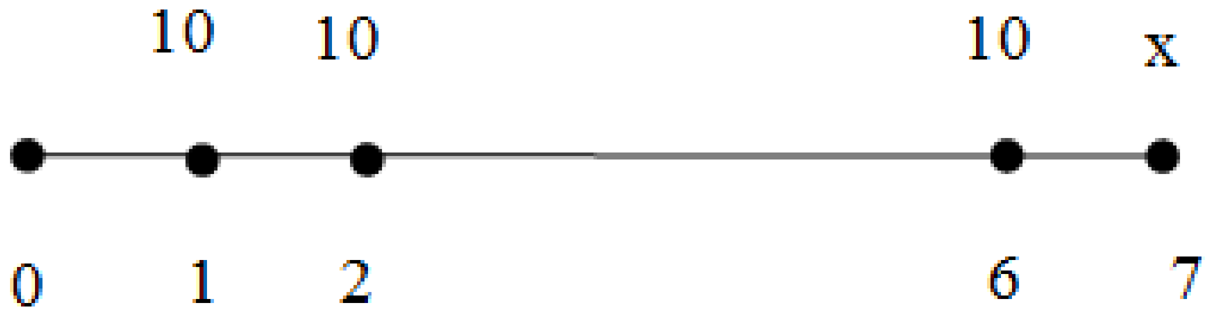
$E = 10$ $K_n = 100$ $r = 12\%$

$$K_n = E \cdot \frac{(1+r)^n - 1}{r}$$

$n = 6.96$

$$n = \frac{\ln\left(\frac{r \cdot K_n}{E} + 1\right)}{\ln(1+r)}$$

Additional payment



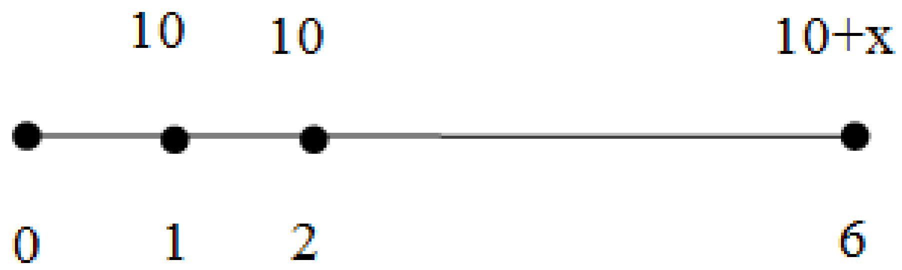
$$K_6 \cdot (1+r) + x = 100$$

$$K_n = E \cdot \frac{(1+r)^n - 1}{r}$$

$$x = 9.11$$

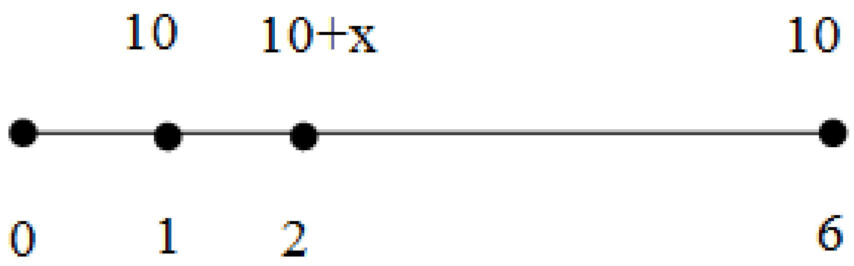
$$K_6 = 81.15$$

Enlargement of one of the payment



$$K_6 + x = 100$$

$$x = 18.85$$



$$K_6 + x \cdot (1 + r)^4 = 100$$

$$x = 11.98$$

New payments

$$E = \frac{r \cdot K_n}{(1+r)^n - 1}$$

$$n = 7 \qquad E = 9.91$$

Example 6c – term of annuity due is not an integer, simple interest

$E = 10$ $K_n = 100$ $r = 12\%$

$$K_n = E \cdot n \cdot \left(1 + \frac{n+1}{2} r\right)$$

$$100 = 10 \cdot n \cdot \left(1 + \frac{n+1}{2} \cdot 0.12\right)$$

$$3 \cdot n^2 + 53 \cdot n - 500 = 0$$

$$\boxed{n = 6.81}$$

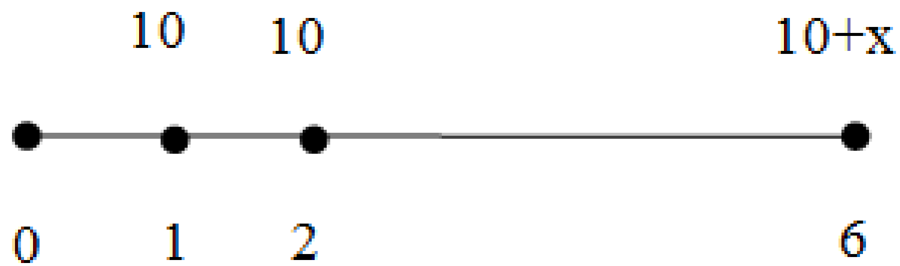
Additional payment

$$K_6 = 10 \cdot 6 \cdot \left(1 + \frac{6+1}{2} \cdot 0.12\right) = 85.2$$

$$K_6 + x = 100 \qquad x = 14.8$$

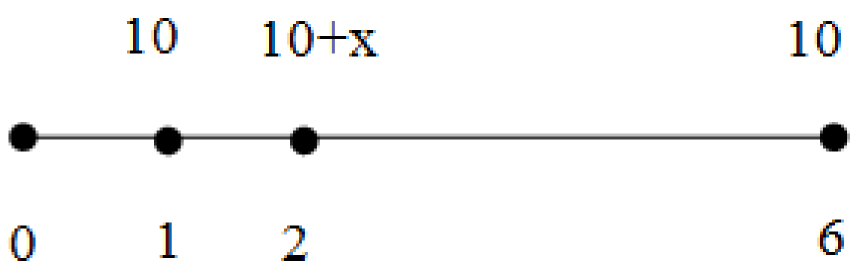
$$(K_6 + x) \cdot (1 + r) = 100 \qquad x = 4.09$$

Enlargement of one of the payment



$$K_6 + x \cdot (1 + r) = 100$$

$$x = 13.21$$



$$K_6 + x \cdot (1 + 5r) = 100$$

$$x = 9.25$$

$$K_6 = 85.2$$

New payments

$$E = \frac{K_n}{n \cdot \left(1 + \frac{n+1}{2} r\right)}$$

$$n = 7 \qquad E = 9.65$$