Fundamentals of Financial Arithmetic

Lecture 5

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- **Long-term loans – repayment methods**

- Equal principal payments per time period
- Equal total payments per time period

- Loan amount – the size or value of the loan
- Interest rate – the annual stated rate of the loan
- Number of payments – the total numbers of payments to pay off the given loan amount
- Payment frequency – loans payments are due monthly (quarterly, annually).
- Compounding coincides with payments (Compounding doesn’t coincide with payments)

- **Loan payment = principal payment + interest payment**

- The amortization schedule shows – for each payment – how much of the payment goes toward the loan principal, and how much is paid on interest.

Example 1 – Loan Amortization Schedule

- An investor borrowed 100 PLN. The loan was for four quarters at 20% annual interest (compounding quarterly).

$$S = 100 \qquad N = 4 \qquad r = \frac{0.2}{4} = 0.05$$

Loan amortization schedule – equal principal payments

(interest payment as a percent of the previous principal balance)

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	100	25	5	30	75
2	75	25	3.75	28.75	50
3	50	25	2.5	27.5	25
4	25	25	1.25	26.25	0
Total		**100**	**12.5**	**112.5**	

	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

Loan amortization schedule – equal principal payments

(interest payment as a percent of the repaid loan)

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	100	25	1.25	26.25	75
2	75	25	2.5	27.5	50
3	50	25	3.75	28.75	25
4	25	25	5	30	0
Total		**100**	**12.5**	**112.5**	

Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

Loan amortization schedule – given principal payments

(interest payment as a percent of the previous principal balance)

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	100	10	5	15	90
2	90	20	4.5	24.5	70
3	70	20	3.5	23.5	50
4	50	50	2.5	52.5	0
Total		**100**	**15.5**	**115.5**	
	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

Loan amortization schedule

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	100	0	5	5	100
2	100	0	5	5	100
3	100	0	5	5	100
4	100	100	5	105	0
Total		**100**	**20**	**120**	

	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

Equal total payments

$$S(1+r)^N = A_1(1+r)^{N-1} + A_2(1+r)^{N-2} + \cdots + A_N$$

$$S = \frac{A_1}{1+r} + \frac{A_2}{(1+r)^2} + \cdots + \frac{A_N}{(1+r)^N}$$

Periodic payment

$$S(1+r)^N = A\frac{(1+r)^N - 1}{r}$$

$$A = \frac{S \cdot r \cdot (1+r)^N}{(1+r)^N - 1}$$

Equal total payments

$$Z_n = r \cdot S_{n-1} \qquad T_n = S_{n-1} - S_n \qquad A_n = T_n + Z_n$$

$$S_n = S(1+r)^n - \left(A_1(1+r)^{n-1} + A_2(1+r)^{n-2} + \cdots + A_{n-1}(1+r)\right) - A_n$$

$$S_n = (1+r)\left(S(1+r)^{n-1} - \left(A_1(1+r)^{n-2} + A_2(1+r)^{n-3} + \cdots + A_{n-1}\right)\right) - A_n$$

$$S_n = (1+r)S_{n-1} - A_n$$

Loan amortization schedule – equal total payments

(interest payment as a percent of the previous principal balance)

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	100	23.2	5	28.2	76.8
2	76.8	24.36	3.84	28.2	52.44
3	52.44	25.58	2.62	28.2	26.86
4	26.86	26.86	1.34	28.2	0
Total		**100**	**12.8**	**112.8**	
	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

Loan amortization schedule – given total payments

(interest payment as a percent of the previous principal balance)

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	100	15	5	20	85
2	85	25.75	4.25	30	59.25
3	59.25	37.04	2.96	40	22.21
4	22.21	22.21	1.11	23.32	0
Total		**100**	**13.32**	**113.32**	

	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

$$S(1+r)^4 = A_1(1+r)^3 + A_2(1+r)^2 + A_3(1+r) + \boxed{A_4}$$

Equal total payments

(continuously compounded interest)

$$Se^{r\cdot N} = A_1 e^{r(N-1)} + A_2 e^{r(N-2)} + \cdots + A_N$$

$$Se^{r\cdot N} = A\frac{e^{r\cdot N}-1}{e^r-1}$$

$$A = S\cdot e^{r\cdot N}\cdot\frac{e^r-1}{e^{r\cdot N}-1}$$

$$Z_n = S_{n-1}\cdot\left(e^r-1\right) \qquad T_n = S_{n-1}-S_n \qquad A_n = T_n + Z_n$$

Loan amortization schedule – equal total payments

(continuously compounded interest)

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	100	23.16	5.13	28.28	76.84
2	76.84	24.34	3.94	28.28	52.5
3	52.5	25.59	2.69	28.28	26.91
4	26.91	26.91	1.38	28.28	0
Total		**100**	**13.14**	**113.14**	

	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

Example 2a – Loan Amortization Schedule

- An investor borrowed 1000 PLN. The loan was for 6 months at 24% annual interest (compound interest rate).
- Create a loan amortization schedule if since the fourth month the annual interest is 18%.

$$S = 1000 \qquad N = 6 \qquad r = \frac{0.24}{12} = 0.02$$

$$A = \frac{S \cdot r \cdot (1+r)^N}{(1+r)^N - 1} \qquad A = \frac{1000 \cdot 0.02 \cdot (1+0.02)^6}{(1+0.02)^6 - 1} = 178.5$$

$$S_3 = 514.8 \qquad N = 3 \qquad r = \frac{0.18}{12} = 0.015$$

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	1000	158.5	20.0	178.5	841.5
2	841.5	161.7	16.8	178.5	679.8
3	679.8	164.9	13.6	178.5	514.8
4	514.8	169.1	7.7	176.8	345.8
5	345.8	171.6	5.2	176.8	174.2
6	174.2	174.2	2.6	176.8	0
Total		**1000**	**65.9**	**1065.9**	
	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

Example 2b – Loan Amortization Schedule

- An investor borrowed 1000 PLN. The loan was for 6 months at 24% annual interest (compound interest rate).
- Create a loan amortization schedule if the investor pays additional 100 PLN with the third payment.

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	1000	158.5	20.0	178.5	841.5
2	841.5	161.7	16.8	178.5	679.8
3	679.8	264.9	13.6	278.5	414.8
4	414.8	170.2	8.3	178.5	244.6
5	244.6	173.6	4.9	178.5	71.0
6	71.0	71.0	1.4	72.4	0.0
Total		**1000**	**65.0**	**1065.0**	
	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

Example 2c – Loan Amortization Schedule

- An investor borrowed 1000 PLN. The loan was for 6 months at 24% annual interest (compound interest rate).
- Create a loan amortization schedule if the investor doesn’t pay the fourth payment. He pays it plus interest with the fifth payment.

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	1000	158.5	20.0	178.5	841.5
2	841.5	161.7	16.8	178.5	679.8
3	679.8	164.9	13.6	178.5	514.8
4	514.8	-10.3	10.3	0	525.1
5	525.1	350.1	10.5	360.6	175.0
6	175.0	175.0	3.5	178.5	0
Total		**1000**	**74.7**	**1074.7**	
	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

Example 2d – Loan Amortization Schedule

- An investor borrowed 1000 PLN. The loan was for 6 months at 24% annual interest (compound interest rate).
- Create a loan amortization schedule if the first payment is postponed for two months.

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	1040.4	164.9	20.8	185.7	875.5
2	875.5	168.2	17.5	185.7	707.2
3	707.2	171.6	14.1	185.7	535.6
4	535.6	175.0	10.7	185.7	360.6
5	360.6	178.5	7.2	185.7	182.1
6	182.1	182.1	3.6	185.7	0
Total		**1040.4**	**74.0**	**1114.4**	
	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

Example 2e – Loan Amortization Schedule

- An investor borrowed 1000 PLN. The loan was for 6 months at 24% annual interest (compound interest rate).
- Create a loan amortization schedule if the investor pays two payments, than he doesn’t pay for 3 months. The investor begins to pay off the loan again in the sixth month paying three equal payments every two months. Since the third month the annual interest rate is 18%.

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	1000	158.5	20.0	178.5	841.5
2	841.5	161.7	16.8	178.5	679.8
6	710.8	237.0	10.7	247.7	473.8
8	480.9	240.5	7.2	247.7	240.4
10	244.0	244.0	3.7	247.7	0
Total		**1000**	**58.4**	**1058.4**	
	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

$$S_5 = 679.8 \cdot (1.015)^3 = 710.8$$

$$S_5 = \frac{A_6}{1+r} + \frac{A_8}{(1+r)^3} + \frac{A_{10}}{(1+r)^5}$$

$$A_6 = A_8 = A_{10} = A$$

$$710.8 = \frac{A}{1.015} + \frac{A}{(1.015)^3} + \frac{A}{(1.015)^5}$$

$$S_7 = 473.8 \cdot 1.015 = 480.9$$

Example 3

- An investor borrowed 50 PLN. Find how many payments of 15 PLN should be made if the effective rate of interest is 10%.
- Solve the problem of non-integer number of payments.

$$S(1+r)^N = A\frac{(1+r)^N - 1}{r}$$

$$S = 50 \qquad A = 15$$

$$r = 0.1$$

$$N = \frac{\ln 1.5}{\ln 1.1} = 4.25$$

	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance
n	S_{n-1}	T_n	Z_n	A_n	S_n
1	50	10	5	15	40
2	40	11	4	15	29
3	29	12.1	2.9	15	16.9
4	16.9	13.3	1.7	15	**3.59**
	3.59	3.59	0.36	3.95	

Additional payment

Enlargement of one of the payment

$$A_1 = A_2 = A_3 = 15 \qquad A_4 = 18.59$$

$$A_2 = A_3 = A_4 = 15 \qquad A_1 = 17.70$$

$$A_1 = A_3 = A_4 = 15 \qquad A_2 = 17.97$$

$$A_1 = A_2 = A_4 = 15 \qquad A_3 = 18.26$$

New payments

$$N = 4$$

$$A = \frac{S \cdot r \cdot (1+r)^N}{(1+r)^N - 1}$$

$$A = 15.77$$