# Fundamentals of Financial Arithmetic
## Lecture 5-6

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- **Long-term loans – repayment methods**

- Equal principal payments per time period
- Equal total payments per time period

- Loan amount – the size or value of the loan
- Interest rate – the annual stated rate of the loan
- Number of payments – the total numbers of payments to pay off the given loan amount
- Payment frequency – loans payments are due monthly (quarterly, annually).
- Compounding coincides with payments (Compounding doesn't coincide with payments)

- **Loan payment = principal payment + interest payment**

- The amortization schedule shows – for each payment – how much of the payment goes toward the loan principal, and how much is paid on interest.

# **Example 1 – Loan Amortization Schedule**

- An investor borrowed 100 PLN. The loan was for four quarters at 20% annual interest (compounding quarterly).

$$S = 100 \qquad\qquad N = 4 \qquad\qquad r = \frac{0.2}{4} = 0.05$$

# Loan amortization schedule – equal principal payments
(interest payment as a percent of the previous principal balance)

| $n$ | $S_{n-1}$ | $T_n$ | $Z_n$ | $A_n$ | $S_n$ |
|---|---|---|---|---|---|
| 1 | 100 | 25 | 5 | 30 | 75 |
| 2 | 75 | 25 | 3.75 | 28.75 | 50 |
| 3 | 50 | 25 | 2.5 | 27.5 | 25 |
| 4 | 25 | 25 | 1.25 | 26.25 | 0 |
| Total | | **100** | **12.5** | **112.5** | |
| | Previous principal balance | Principal payment | Interest payment | Total payment | Principal balance |

# Loan amortization schedule – equal principal payments
### (interest payment as a percent of the repaid loan)

| $n$ | $S_{n-1}$ | $T_n$ | $Z_n$ | $A_n$ | $S_n$ |
|---|---|---|---|---|---|
| 1 | 100 | 25 | 1.25 | 26.25 | 75 |
| 2 | 75 | 25 | 2.5 | 27.5 | 50 |
| 3 | 50 | 25 | 3.75 | 28.75 | 25 |
| 4 | 25 | 25 | 5 | 30 | 0 |
| Total | | **100** | **12.5** | **112.5** | |

| Previous principal balance | Principal payment | Interest payment | Total payment | Principal balance |

# Loan amortization schedule – given principal payments
(interest payment as a percent of the previous principal balance)

| $n$ | $S_{n-1}$ | $T_n$ | $Z_n$ | $A_n$ | $S_n$ |
|-----|-----------|-------|-------|-------|-------|
| 1 | 100 | 10 | 5 | 15 | 90 |
| 2 | 90 | 20 | 4.5 | 24.5 | 70 |
| 3 | 70 | 20 | 3.5 | 23.5 | 50 |
| 4 | 50 | 50 | 2.5 | 52.5 | 0 |
| Total | | **100** | **15.5** | **115.5** | |
| | Previous principal balance | Principal payment | Interest payment | Total payment | Principal balance |

# Loan amortization schedule

| $n$ | $S_{n-1}$ | $T_n$ | $Z_n$ | $A_n$ | $S_n$ |
|---|---|---|---|---|---|
| 1 | 100 | 0 | 5 | 5 | 100 |
| 2 | 100 | 0 | 5 | 5 | 100 |
| 3 | 100 | 0 | 5 | 5 | 100 |
| 4 | 100 | 100 | 5 | 105 | 0 |
| Total | | **100** | **20** | **120** | |

| Previous principal balance | Principal payment | Interest payment | Total payment | Principal balance |

# Equal total payments

$$S(1+r)^N = A_1(1+r)^{N-1} + A_2(1+r)^{N-2} + \cdots + A_N$$

$$S = \frac{A_1}{1+r} + \frac{A_2}{(1+r)^2} + \cdots + \frac{A_N}{(1+r)^N}$$

Periodic payment

$$S(1+r)^N = A\frac{(1+r)^N - 1}{r}$$

$$\boxed{A = \frac{S \cdot r \cdot (1+r)^N}{(1+r)^N - 1}}$$

# Equal total payments

$$Z_n = r \cdot S_{n-1} \qquad T_n = S_{n-1} - S_n \qquad A_n = T_n + Z_n$$

$$S_n = S(1+r)^n - \left( A_1\left(1+r\right)^{n-1} + A_2\left(1+r\right)^{n-2} + \cdots + A_{n-1}(1+r) \right) - A_n$$

$$S_n = (1+r)\left( S(1+r)^{n-1} - \left( A_1\left(1+r\right)^{n-2} + A_2\left(1+r\right)^{n-3} + \cdots + A_{n-1} \right) \right) - A_n$$

$$S_n = (1+r)S_{n-1} - A_n$$

# Loan amortization schedule – equal total payments
(interest payment as a percent of the previous principal balance)

| $n$ | $S_{n-1}$ | $T_n$ | $Z_n$ | $A_n$ | $S_n$ |
|---|---|---|---|---|---|
| 1 | 100 | 23.2 | 5 | 28.2 | 76.8 |
| 2 | 76.8 | 24.36 | 3.84 | 28.2 | 52.44 |
| 3 | 52.44 | 25.58 | 2.62 | 28.2 | 26.86 |
| 4 | 26.86 | 26.86 | 1.34 | 28.2 | 0 |
| Total | | **100** | **12.8** | **112.8** | |
| | Previous principal balance | Principal payment | Interest payment | Total payment | Principal balance |

# Loan amortization schedule – given total payments
(interest payment as a percent of the previous principal balance)

| $n$ | $S_{n-1}$ | $T_n$ | $Z_n$ | $A_n$ | $S_n$ |
|---|---|---|---|---|---|
| 1 | 100 | 15 | 5 | 20 | 85 |
| 2 | 85 | 25.75 | 4.25 | 30 | 59.25 |
| 3 | 59.25 | 37.04 | 2.96 | 40 | 22.21 |
| 4 | 22.21 | 22.21 | 1.11 | 23.32 | 0 |
| Total | | **100** | **13.32** | **113.32** | |

| Previous principal balance | Principal payment | Interest payment | Total payment | Principal balance |

$$S(1+r)^4 = A_1(1+r)^3 + A_2(1+r)^2 + A_3(1+r) + \boxed{A_4}$$

# Equal total payments
## (continuously compounded interest)

$$Se^{\,r\cdot N} = A_1 e^{r(N-1)} + A_2 e^{r(N-2)} + \cdots + A_N$$

$$Se^{\,r\cdot N} = A\,\frac{e^{r\cdot N}-1}{e^{r}-1}$$

$$A = S\cdot e^{\,r\cdot N}\cdot\frac{e^{r}-1}{e^{r\cdot N}-1}$$

$$Z_n = S_{n-1}\cdot\left(e^{r}-1\right) \qquad T_n = S_{n-1} - S_n \qquad A_n = T_n + Z_n$$

# Loan amortization schedule – equal total payments
## (continuously compounded interest)

| $n$ | $S_{n-1}$ | $T_n$ | $Z_n$ | $A_n$ | $S_n$ |
|---|---|---|---|---|---|
| 1 | 100 | 23.16 | 5.13 | 28.28 | 76.84 |
| 2 | 76.84 | 24.34 | 3.94 | 28.28 | 52.5 |
| 3 | 52.5 | 25.59 | 2.69 | 28.28 | 26.91 |
| 4 | 26.91 | 26.91 | 1.38 | 28.28 | 0 |
| Total | | **100** | **13.14** | **113.14** | |
| | Previous principal balance | Principal payment | Interest payment | Total payment | Principal balance |

# Example 1a – Loan Amortization Schedule

- An investor borrowed 1000 PLN. The loan was for 6 months at 24% annual interest (compound interest rate).

- Create a loan amortization schedule if since the fourth month the annual interest is 18%.

$$S = 1000 \qquad N = 6 \qquad r = \frac{0.24}{12} = 0.02$$

$$A = \frac{S \cdot r \cdot (1+r)^N}{(1+r)^N - 1} \qquad A = \frac{1000 \cdot 0.02 \cdot (1+0.02)^6}{(1+0.02)^6 - 1} = 178.5$$

$$S_3 = 514.8 \qquad N = 3 \qquad r = \frac{0.18}{12} = 0.015$$

| $n$ | $S_{n-1}$ | $T_n$ | $Z_n$ | $A_n$ | $S_n$ |
|---|---|---|---|---|---|
| 1 | 1000 | 158.5 | 20.0 | 178.5 | 841.5 |
| 2 | 841.5 | 161.7 | 16.8 | 178.5 | 679.8 |
| 3 | 679.8 | 164.9 | 13.6 | 178.5 | 514.8 |
| 4 | 514.8 | 169.1 | 7.7 | 176.8 | 345.8 |
| 5 | 345.8 | 171.6 | 5.2 | 176.8 | 174.2 |
| 6 | 174.2 | 174.2 | 2.6 | 176.8 | 0 |
| Total | | **1000** | **65.9** | **1065.9** | |
| | Previous principal balance | Principal payment | Interest payment | Total payment | Principal balance |

# Example 1b – Loan Amortization Schedule

- An investor borrowed 1000 PLN. The loan was for 6 months at 24% annual interest (compound interest rate).

- Create a loan amortization schedule if the investor pays additional 100 PLN with the third payment.

| $n$ | $S_{n-1}$ | $T_n$ | $Z_n$ | $A_n$ | $S_n$ |
|---|---|---|---|---|---|
| 1 | 1000 | 158.5 | 20.0 | 178.5 | 841.5 |
| 2 | 841.5 | 161.7 | 16.8 | 178.5 | 679.8 |
| 3 | 679.8 | 264.9 | 13.6 | 278.5 | 414.8 |
| 4 | 414.8 | 170.2 | 8.3 | 178.5 | 244.6 |
| 5 | 244.6 | 173.6 | 4.9 | 178.5 | 71.0 |
| 6 | 71.0 | 71.0 | 1.4 | 72.4 | 0.0 |
| Total | | **1000** | **65.0** | **1065.0** | |

| | Previous principal balance | Principal payment | Interest payment | Total payment | Principal balance |
|---|---|---|---|---|---|

# Example 1c – Loan Amortization Schedule

- An investor borrowed 1000 PLN. The loan was for 6 months at 24% annual interest (compound interest rate).

- Create a loan amortization schedule if the investor doesn't pay the fourth payment. He pays it plus interest with the fifth payment.

| $n$ | $S_{n-1}$ | $T_n$ | $Z_n$ | $A_n$ | $S_n$ |
|---|---|---|---|---|---|
| 1 | 1000 | 158.5 | 20.0 | 178.5 | 841.5 |
| 2 | 841.5 | 161.7 | 16.8 | 178.5 | 679.8 |
| 3 | 679.8 | 164.9 | 13.6 | 178.5 | 514.8 |
| 4 | 514.8 | -10.3 | 10.3 | 0 | 525.1 |
| 5 | 525.1 | 350.1 | 10.5 | 360.6 | 175.0 |
| 6 | 175.0 | 175.0 | 3.5 | 178.5 | 0 |
| Total | | **1000** | **74.7** | **1074.7** | |
| | Previous principal balance | Principal payment | Interest payment | Total payment | Principal balance |

# Example 1d – Loan Amortization Schedule

- An investor borrowed 1000 PLN. The loan was for 6 months at 24% annual interest (compound interest rate).

- Create a loan amortization schedule if the first payment is postponed for two months.

| $n$ | $S_{n-1}$ | $T_n$ | $Z_n$ | $A_n$ | $S_n$ |
|-----|-----------|-------|-------|-------|-------|
| 1 | 1040.4 | 164.9 | 20.8 | 185.7 | 875.5 |
| 2 | 875.5 | 168.2 | 17.5 | 185.7 | 707.2 |
| 3 | 707.2 | 171.6 | 14.1 | 185.7 | 535.6 |
| 4 | 535.6 | 175.0 | 10.7 | 185.7 | 360.6 |
| 5 | 360.6 | 178.5 | 7.2 | 185.7 | 182.1 |
| 6 | 182.1 | 182.1 | 3.6 | 185.7 | 0 |
| Total | | **1040.4** | **74.0** | **1114.4** | |

| Previous principal balance | Principal payment | Interest payment | Total payment | Principal balance |

# Example 1e – Loan Amortization Schedule

- An investor borrowed 1000 PLN. The loan was for 6 months at 24% annual interest (compound interest rate).

- Create a loan amortization schedule if the investor pays two payments, than he doesn't pay for 3 months. The investor begins to pay off the loan again in the sixth month paying three equal payments every two months. Since the third month the annual interest rate is 18%.

| $n$ | $S_{n-1}$ | $T_n$ | $Z_n$ | $A_n$ | $S_n$ |
|---|---|---|---|---|---|
| 1 | 1000 | 158.5 | 20.0 | 178.5 | 841.5 |
| 2 | 841.5 | 161.7 | 16.8 | 178.5 | 679.8 |
| 6 | 710.8 | 237.0 | 10.7 | 247.7 | 473.8 |
| 8 | 480.9 | 240.5 | 7.2 | 247.7 | 240.4 |
| 10 | 244.0 | 244.0 | 3.7 | 247.7 | 0 |
| Total | | **1000** | **58.4** | **1058.4** | |

| | Previous principal balance | Principal payment | Interest payment | Total payment | Principal balance |

$$S_5 = 679.8 \cdot (1.015)^3 = 710.8$$

$$S_5 = \frac{A_6}{1+r} + \frac{A_8}{(1+r)^3} + \frac{A_{10}}{(1+r)^5}$$

$$A_6 = A_8 = A_{10} = A$$

$$710.8 = \frac{A}{1.015} + \frac{A}{(1.015)^3} + \frac{A}{(1.015)^5}$$

$$S_7 = 473.8 \cdot 1.015 = 480.9$$

# Example 2

- An investor borrowed 50 PLN. Find how many payments of 15 PLN should be made if the effective rate of interest is 10%.

- Solve the problem of non-integer number of payments.

$$S(1+r)^N = A\frac{(1+r)^N - 1}{r}$$

$$S = 50 \qquad A = 15$$

$$r = 0.1$$

$$N = \frac{\ln 1.5}{\ln 1.1} = 4.25$$

| $n$ | Previous principal balance $S_{n-1}$ | Principal payment $T_n$ | Interest payment $Z_n$ | Total payment $A_n$ | Principal balance $S_n$ |
|---|---|---|---|---|---|
| 1 | 50 | 10 | 5 | 15 | 40 |
| 2 | 40 | 11 | 4 | 15 | 29 |
| 3 | 29 | 12.1 | 2.9 | 15 | 16.9 |
| 4 | 16.9 | 13.3 | 1.7 | 15 | **3.59** |
|  | 3.59 | 3.59 | 0.36 | 3.95 |  |

Additional payment

# Enlargement of one of the payment

$$A_1 = A_2 = A_3 = 15$$

$$A_4 = 18.59$$

$$A_2 = A_3 = A_4 = 15$$

$$A_1 = 17.70$$

$$A_1 = A_3 = A_4 = 15$$

$$A_2 = 17.97$$

$$A_1 = A_2 = A_4 = 15$$

$$A_3 = 18.26$$

# New payments

$$N = 4$$

$$A = \frac{S \cdot r \cdot (1+r)^N}{(1+r)^N - 1}$$

$$A = 15.77$$

# **Example 3 – Debt consolidation loans**

- 12 monthly payments of 10 PLN, 15% annual interest rate (compounding quarterly)

- 5 semi-annual payments of 100 PLN, 12% annual interest rate (compounding monthly).

- 10 quarterly payments of consolidated loan, 18% annual interest rate (compounding annually)

# Example – Debt consolidation loans

$$S = \frac{A}{(1+r)^N} \frac{(1+r)^N - 1}{r}$$

$$S_1 = \frac{10}{(1+r_1)^{12}} \frac{(1+r_1)^{12} - 1}{r_1} \qquad r_1 = \left(1 + \frac{0.15}{4}\right)^{\frac{1}{3}} = 0.01235$$

$$S_2 = \frac{100}{(1+r_2)^{5}} \frac{(1+r_2)^{5} - 1}{r_2} \qquad r_2 = \left(1 + \frac{0.12}{12}\right)^{6} = 0.06152$$

$$S_1 = 110.9 \qquad\qquad S_2 = 419.5$$

# Example – Debt consolidation loans

$$S = 530.4$$

$$A = \frac{S \cdot r \cdot (1+r)^N}{(1+r)^N - 1}$$

$$r = (1+0.18)^{\frac{1}{4}} = 0.04225$$

$$A = 66.1$$