Fundamentals of Financial Arithmetic

Lecture 6

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Example – Debt consolidation loans

- 12 monthly payments of 10 PLN, 15% annual interest rate (compounding quarterly)
- 5 semi-annual payments of 100 PLN, 12% annual interest rate (compounding monthly).

- 10 quarterly payments of consolidated loan, 18% annual interest rate (compounding annually)

Example – Debt consolidation loans

$$S = \frac{A}{(1+r)^N} \frac{(1+r)^N - 1}{r}$$

$$S_1 = \frac{10}{(1+r_1)^{12}} \frac{(1+r_1)^{12} - 1}{r_1} \qquad r_1 = \left(1 + \frac{0.15}{4}\right)^{\frac{1}{3}} = 0.01235$$

$$S_2 = \frac{100}{(1+r_2)^5} \frac{(1+r_2)^5 - 1}{r_2} \qquad r_2 = \left(1 + \frac{0.12}{12}\right)^6 = 0.06152$$

$$S_1 = 110.9 \qquad S_2 = 419.5$$

Example – Debt consolidation loans

$$S = 530.4$$

$$A = \frac{S \cdot r \cdot (1+r)^N}{(1+r)^N - 1}$$

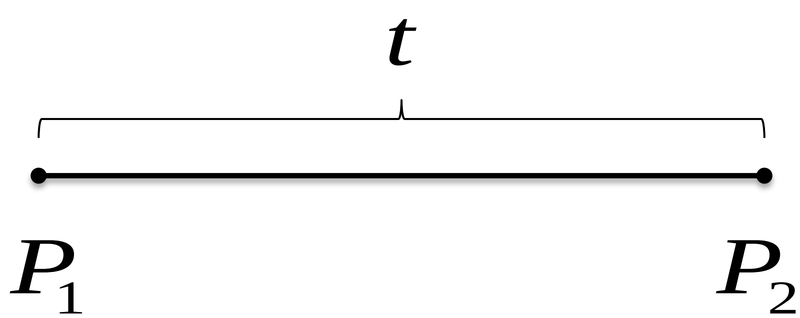
$$r = (1 + 0.18)^{\frac{1}{4}} = 0.04225$$

$$A = 66.1$$

Treasury bills

- Treasury bills are discounted short-term debt securities with maturities of up to one year.
- Treasury bills are sold at a discount off their nominal value.
- Treasury bills represent an important instrument of governmental fiscal policy and the central bank's monetary policy.
- The nominal value is payable to the final holder upon redemption on maturity.
- Nominal/face value – 10 000 PLN in Poland.
- Maturity – the date the bill is redeemed and the investor is paid the face value amount.
- Regular Treasury bill series are issued weekly (13, 26 or 52 weeks in Poland).

Bill valuation methods

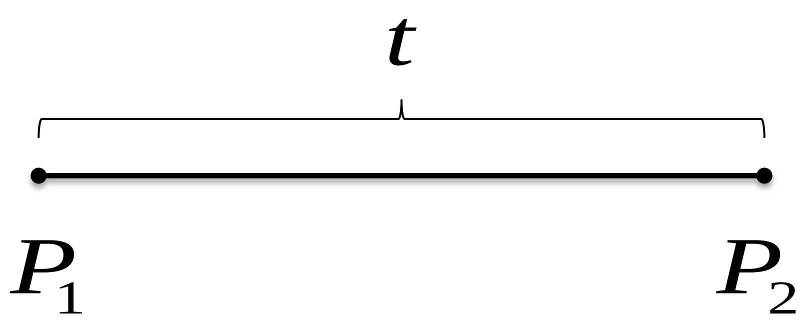


- P_1 – purchase price (at which investor can buy)
- P_2 – nominal/face value (principal)
- t – number o days from purchase to maturity

Bill valuation methods

- The method applied to determine the value of bills depends on whether the bill price is based on the rate of return (r) or the rate of discount (d).
- Bond prices are quoted relative to a 100 PLN face/nominal value.

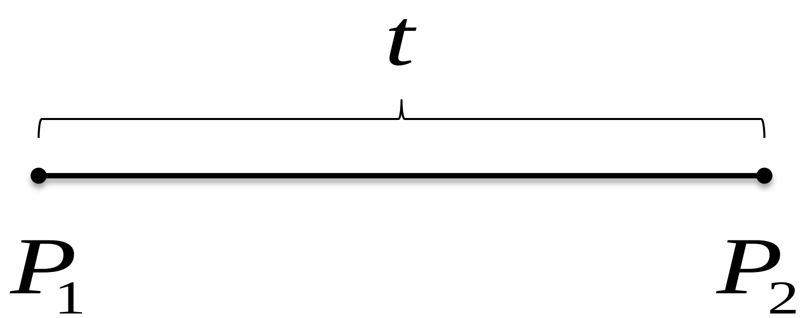
Treasury bills – the rate of return



$$\frac{P_2 - P_1}{P_1} \rightarrow t$$

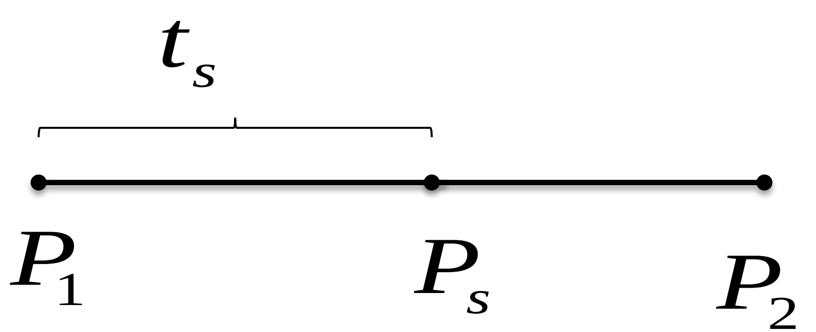
$$r \rightarrow 360$$

Treasury bills – the rate of return



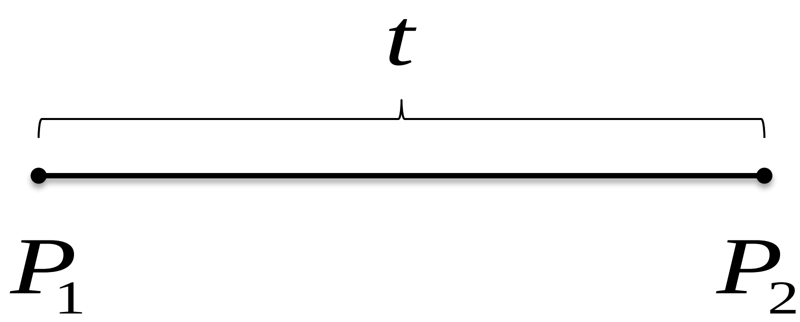
$$r = \frac{P_2 - P_1}{P_1} \cdot \frac{360}{t}$$

Treasury bills – the rate of return for the holding period



$$r_s = \frac{P_s - P_1}{P_1} \cdot \frac{360}{t_s}$$

Treasury bills – the discount rate



$$d = \frac{P_2 - P_1}{P_2} \cdot \frac{360}{t}$$

Treasury bills – price of the Treasury bills

- The price per 100 PLN principal (bills quoted on the basis of the rate of return).

$$P = \frac{360}{r \cdot t + 360} \cdot 100$$

- The price per 100 PLN principal (bills quoted on the basis of the discount rate)

$$P = \left(1 - \frac{d \cdot t}{360}\right) \cdot 100$$

Treasury bills

$$\frac{360}{r \cdot t + 360} \cdot 100 = \left(1 - \frac{d \cdot t}{360}\right) \cdot 100$$

$$r = \frac{d}{1 - d \cdot \frac{t}{360}}$$

$$d = \frac{r}{1 + r \cdot \frac{t}{360}}$$

The rate of return for the known discount rate

The discount rate for the known rate of return

Example 1 – Treasury bills

Investor buys Treasury bills at the primary market with maturity 26 weeks. The nominal value of bills is 1.5 million PLN. The investors pays 97.9005 per a 100 PLN.

$$9790.05 \cdot 150 = 1468508$$

- The rate of return

$$r = \frac{100 - 97.9005}{97.9005} \cdot \frac{360}{182} = 0.04242$$

- The discount rate

$$d = \frac{100 - 97.9005}{100} \cdot \frac{360}{182} = 0.04153$$

Example 2 – Treasury bills

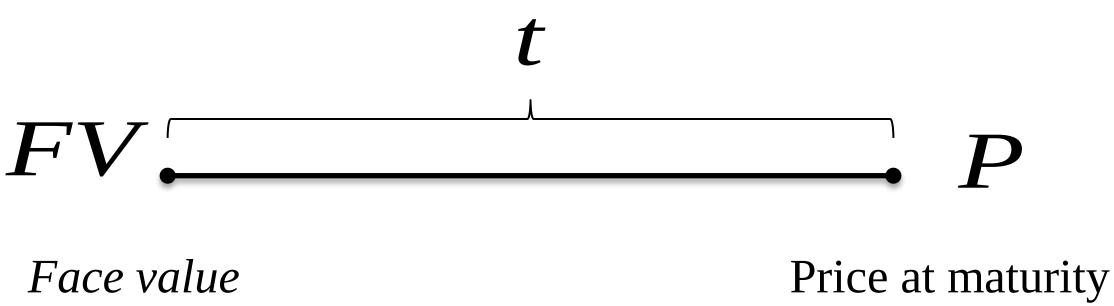
- Assuming that the Treasury bills have been issued at a rate of return of 9% per 60 days, calculate the appropriate discount rate.

$$d = \frac{r}{1 + r \cdot \frac{t}{360}} = \frac{0.09}{1 + 0.09 \cdot \frac{60}{360}} = 0.08867$$

A certificate of deposit – CD

- A certificate of deposit is a savings certificate with a fixed maturity date, specified fixed interest rate issued by commercial banks.
- A CD restricts access to the funds until the maturity date of the investment.

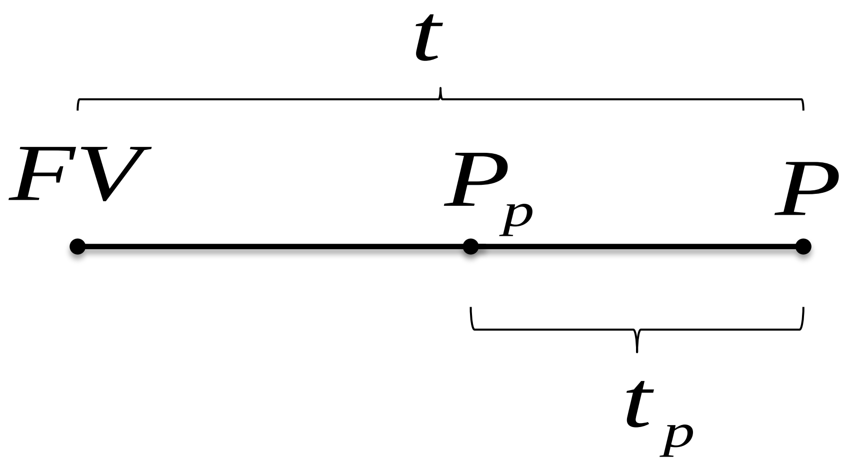
A certificate of deposit



$$P = FV \cdot \left(1 + r_k \cdot \frac{t}{360}\right)$$

r_k – interest rate

A certificate of deposit



Number of days
from purchase to maturity

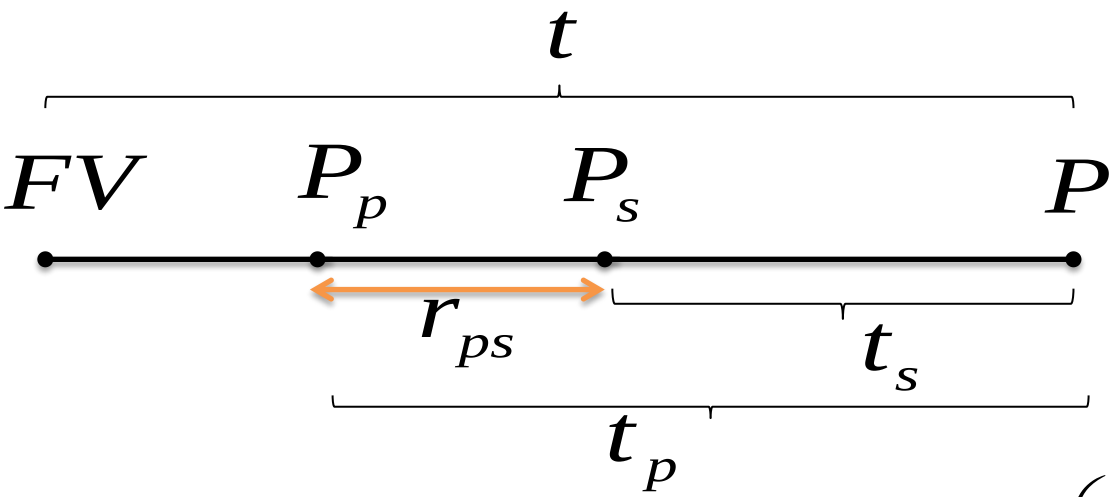
$$FV \cdot \left(1 + r_k \cdot \frac{t}{360}\right) = P_p \cdot \left(1 + r_p \cdot \frac{t_p}{360}\right)$$

$$P_p = \frac{FV \cdot \left(1 + r_k \cdot \frac{t}{360}\right)}{\left(1 + r_p \cdot \frac{t_p}{360}\right)}$$

$$P_p = \frac{100 \cdot \left(1 + r_k \cdot \frac{t}{360}\right)}{\left(1 + r_p \cdot \frac{t_p}{360}\right)}$$

Purchase price

CD – the rate of return for the holding period



$$r_{ps} = \frac{P_s - P_p}{P_p} \cdot \frac{360}{t_p - t_s}$$

Purchase price

$$P_p = \frac{100 \cdot \left(1 + r_k \cdot \frac{t}{360}\right)}{\left(1 + r_p \cdot \frac{t_p}{360}\right)}$$

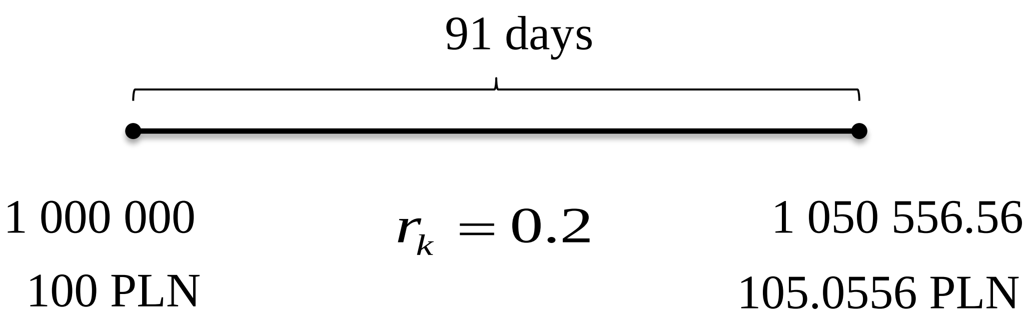
$$r_{ps} = \left(\frac{1 + r_p \cdot \frac{t_p}{360}}{1 + r_s \cdot \frac{t_s}{360}} - 1 \right) \cdot \frac{360}{t_p - t_s}$$

$$P_s = \frac{100 \cdot \left(1 + r_k \cdot \frac{t}{360}\right)}{\left(1 + r_s \cdot \frac{t_s}{360}\right)}$$

Sell price

Example 3 – CD

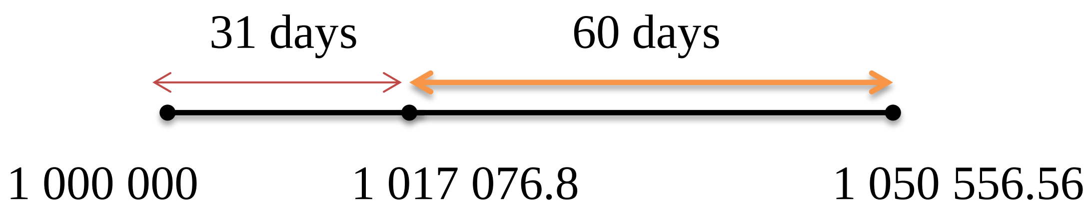
- Investor buys CD at the primary market with maturity 13 weeks. The nominal value of CD is 1 million PLN. The rate of return is 20%.
- Calculate the price at maturity



$$P = 1000000 \cdot \left(1 + 0.2 \cdot \frac{91}{360}\right) = 1050556.556$$

Example 3 – CD

- After 31 days the investor sells CD at a 19.75% rate of return.



$$P_s = \frac{1000000 \cdot \left(1 + 0.2 \cdot \frac{91}{360}\right)}{\left(1 + 0.1975 \cdot \frac{60}{360}\right)} = 1017076.8$$

101.7077 – dirty price

101.7077-1.7222 = 99.9855 – clean price

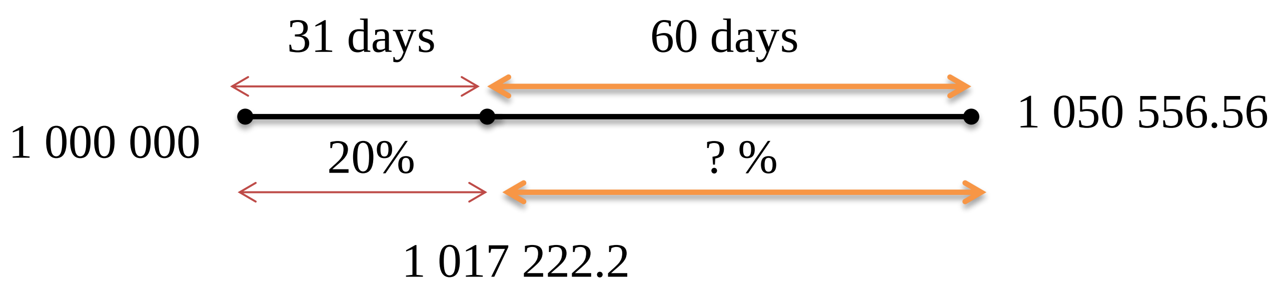
Interest for 100 PLN

$$100 \cdot \frac{0.2 \cdot 31}{360} = 1.722$$

1 017 076.8-17 222.2= 999 854.6

-145,4 PLN

Example 3 – CD



$$P_s = 1000000 \cdot \left(1 + 0.2 \cdot \frac{31}{360}\right) = 1017222.2$$

$$r_s = \frac{1050556.56 - 1017222.2}{1017222.2} \cdot \frac{360}{60} = \boxed{0.1966}$$

Fundamentals of bond valuation

- Bond – a loan between a borrower (issuer) and a lender (investor, creditor)
- The issuer promises to make regular interest payments to the investor at a specified rate (the **coupon rate**) on the amount it has borrowed (the **face/par amount**) until a specified date (the **maturity date**).
- Once the bond matures, the interest payments stop and the issuer is required to repay the face amount of the principal to the investor.

Fundamentals of bond valuation

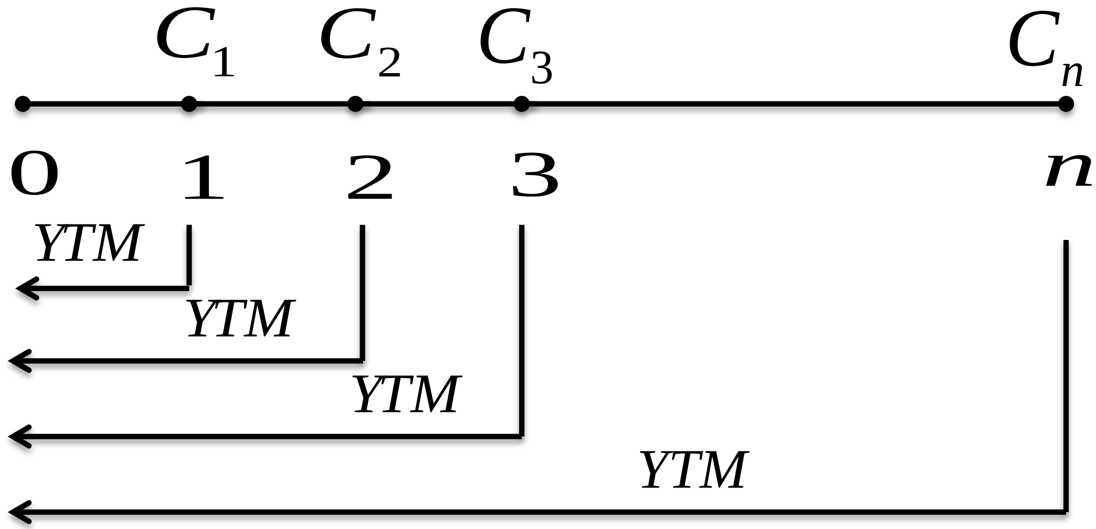
- Bonds can be priced at **a premium**, **discount**, or **at par**.
- If the bond's price is higher than its par value, it will sell at a premium because its interest rate is higher than current prevailing rates.
- If the bond's price is lower than its par value, the bond will sell at a discount because its interest rate is lower than current prevailing interest rates.

Fundamentals of bond valuation

- Bond valuation is the determination of the fair price of a bond.
- The price of bond is the sum of the present values of all expected coupon payments plus the present value of the par value at maturity.
- Yield to maturity – is the internal rate of return earned by investor who buys the bond today at the market price, assuming that the bond will be held until maturity.

Bond pricing – coupon bonds

- C_i – income from the ownership bonds in time i, n – number of payments, YTM – yield to maturity, P – bond price



$$P = \frac{C_1}{1+YTM} + \frac{C_2}{(1+YTM)^2} + \cdots + \frac{C_n}{(1+YTM)^n} = \sum_{i=1}^{n} \frac{C_i}{(1+YTM)^i}$$

Bond pricing – coupon bonds

- **Constant coupon rate**, C – coupon payment, M – value at maturity or par value, n – number of payments, YTM – yield to maturity, P – bond price

$$P = \frac{C}{1+YTM} + \frac{C}{(1+YTM)^2} + \cdots + \frac{C+M}{(1+YTM)^n}$$

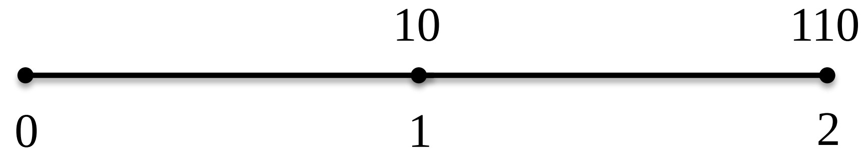
$$P = \frac{C}{1+YTM}\left(1 + \frac{1}{1+YTM} + \cdots + \frac{1}{(1+YTM)^{n-1}}\right) + \frac{M}{(1+YTM)^n}$$

$$P = C \cdot \frac{1-(1+YTM)^{-n}}{YTM} + \frac{M}{(1+YTM)^n}$$

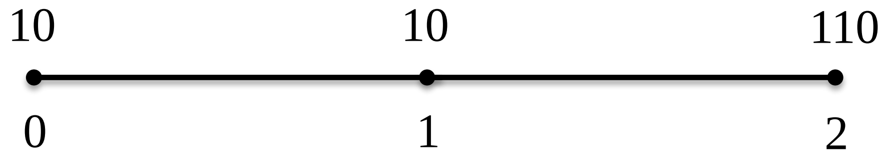
Example 4 – coupon bond

- Calculate the price of a bond with a par value of 100 PLN to be paid in two years (after and before the coupon payment), a coupon rate of 10%, and a required yield of 9%.

$$P = \frac{10}{1.09} + \frac{110}{(1.09)^2} = 101.76$$



$$P = 10 + \frac{10}{1.09} + \frac{110}{(1.09)^2} = 111.76$$



Example 5 – coupon bond

- Calculate the price of a bond with a par value of 100 PLN to be paid in two years and six months, a coupon rate of 10%, and a required yield of 8%. An annual coupon payment.

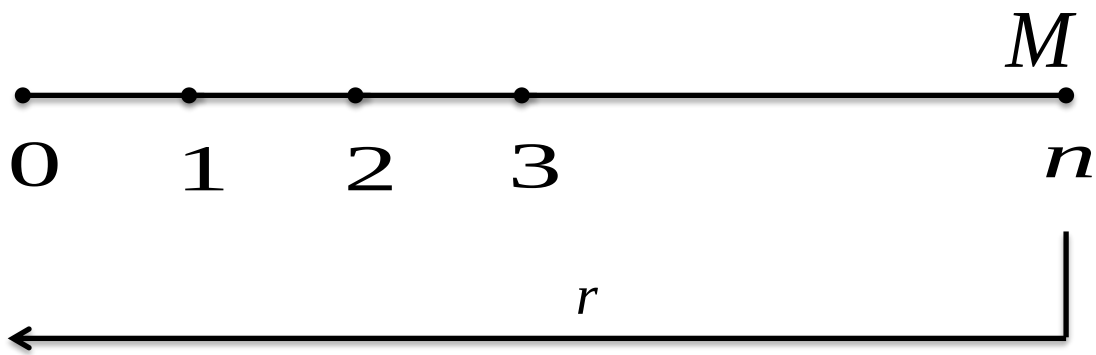
$$P = \frac{10}{(1.08)^{0.5}} + \frac{10}{(1.08)^{1.5}} + \frac{110}{(1.08)^{2.5}} = 109.28$$

Zero-coupon bonds

- Zero-coupon or accrual bonds do not pay a coupon. Instead, these types of bonds are issued at a deep discount and pay the full face value at maturity.

Fundamentals of bond valuation – bond price

- **Zero-coupon bond**, M – value at maturity, n – number of periods, r – interest rate, P – bond price



$$P = \frac{M}{(1+r)^n}$$

Example 6 – pricing zero-coupon bonds

- Calculate the price of a zero-coupon bond that is maturing in one and a half years, has a par value of 100 PLN and a required yield of 5%.

$$P = \frac{100}{(1+0.05)^{1.5}} = 92.94$$

Perpetual bond – pricing

- A bond with no maturity date. Issuers pay coupons forever.

$$P = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \quad \cdots$$

$$P = \frac{C}{r}$$

- C – coupon interest on bond, *r* – an expected yield for maximum term available