Financial Mathematics

Lecture 10

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Example

- Create a three-period binominal price tree and find the fair value of an European call and put options and an American put option on a nondividend-paying stock if the initial stock price is 62 PLN, the compound risk-free interest rate is 12% per annum, the stock volatility is 20%, the strike price of 60 PLN is expiring at the end of the third month (at the end of the third week).

Period - a month

$$u = e^{0.1\cdot\sqrt{1/12}} = 1.029288$$

$$d = 1/u = 0.971545$$

$$R = 1 + 0.12/12 = 1.01$$

$$q = 0.665965$$

$$1 - q = 0.334035$$

Period - a week

$$u = e^{0.1\cdot\sqrt{1/52}} = 1.013964$$

$$d = 1/u = 0.986228$$

$$R = 1 + 0.12/52 = 1.002308$$

$$q = 0.579736$$

$$1 - q = 0.420264$$

Three months

Price tree

62.00	63.82	65.68	67.61
	60.24	62.00	63.82
		58.52	60.24
			56.86

European call option

3.88	5.00	6.28	7.61
	1.76	2.59	3.82
		0.16	0.24
			0.00

European put option

0.11	0.00	0.00	0.00
	0.34	0.00	0.00
		1.04	0.00
			3.14

American put option

0.16	0.00	0.00	0.00
	0.49	0.00	0.00
		1.48	0.00
			3.14

Period - three weeks

Price tree

62.00	62.87	63.74	64.63
	61.15	62.00	62.87
		60.30	61.15
			59.47

European call option

2.45	3.14	3.88	4.63
	1.51	2.14	2.87
		0.66	1.15
			0.00

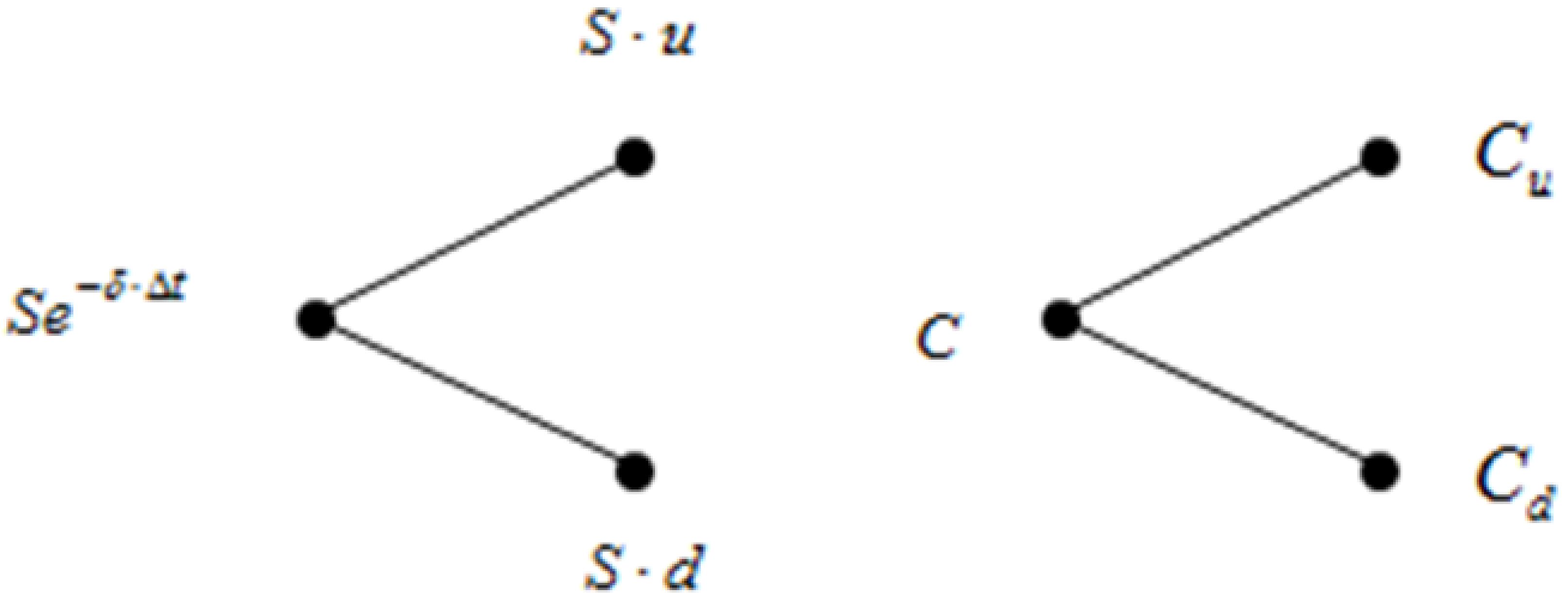
European put option

0.04	0.00	0.00	0.00
	0.09	0.00	0.00
		0.22	0.00
			0.53

American put option

0.04	0.00	0.00	0.00
	0.09	0.00	0.00
		0.22	0.00
			0.53

The underlying asset pays continuous dividend δ



$$u \cdot x + b \cdot e^{r \cdot \Delta t} = C_u$$

$$x = \frac{C_u - C_d}{u - d}$$

$$d \cdot x + b \cdot e^{r \cdot \Delta t} = C_d$$

$$b = e^{-r \cdot \Delta t}\left(\frac{u \cdot C_d - d \cdot C_u}{u - d}\right)$$

$$C = x \cdot e^{-\delta \cdot \Delta t} + b = \left(\frac{C_u - C_d}{u - d}\right) e^{-\delta \cdot \Delta t} + \left(\frac{u \cdot C_d - d \cdot C_u}{u - d}\right) e^{-r \cdot \Delta t}$$

$$C = x \cdot e^{-\delta \cdot \Delta t} + b = \left(\frac{C_u - C_d}{u - d} \right) e^{-\delta \cdot \Delta t} + \left(\frac{u \cdot C_d - d \cdot C_u}{u - d} \right) e^{-r \cdot \Delta t}$$

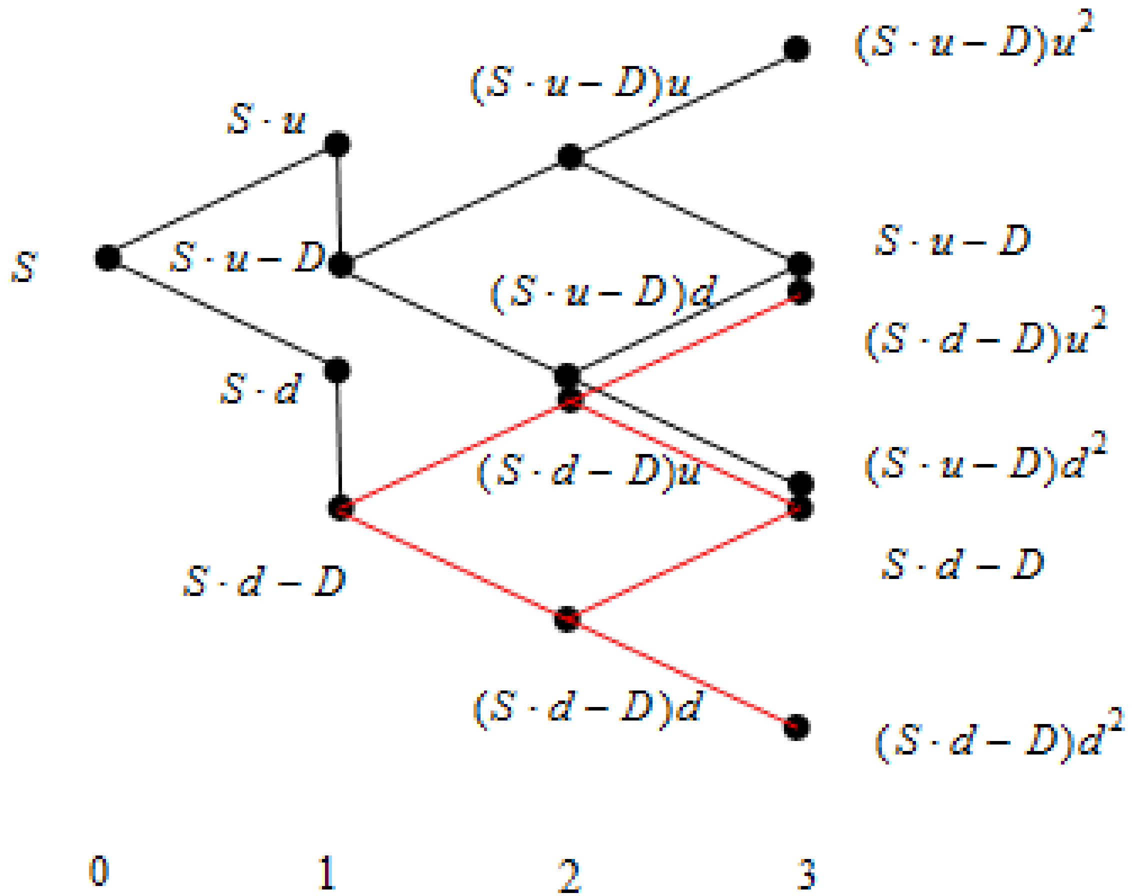
$$C = e^{-r \cdot \Delta t} \left[\left(\frac{C_u - C_d}{u - d} \right) e^{(r-\delta) \cdot \Delta t} + \left(\frac{u \cdot C_d - d \cdot C_u}{u - d} \right) \right]$$

$$C = e^{-r \cdot \Delta t} \left[\frac{e^{(r-\delta) \cdot \Delta t} - d}{u - d} C_u + \frac{u - e^{(r-\delta) \cdot \Delta t}}{u - d} C_d \right]$$

$$C = e^{-r \cdot \Delta t} \left(q \cdot C_u + (1 - q) \cdot C_d \right)$$

$$q = \frac{e^{(r-\delta) \cdot \Delta t} - d}{u - d}$$

Incoherent binomial option tree
(the underlying asset pays predictable income)



Example

- Find the fair value of an European call option using the incoherent binomial option tree if the underlying asset pays dividend of 2 PLN in half a month. The initial stock price is 50 PLN, the strike price of 48 PLN is expiring at the end of the third month, the continuously compounded risk-free interest rate is 10% per annum, and the stock volatility is 20%

$$u = e^{0.2\cdot\sqrt{1/12}} = 1.059434$$

$$q = 0.558$$

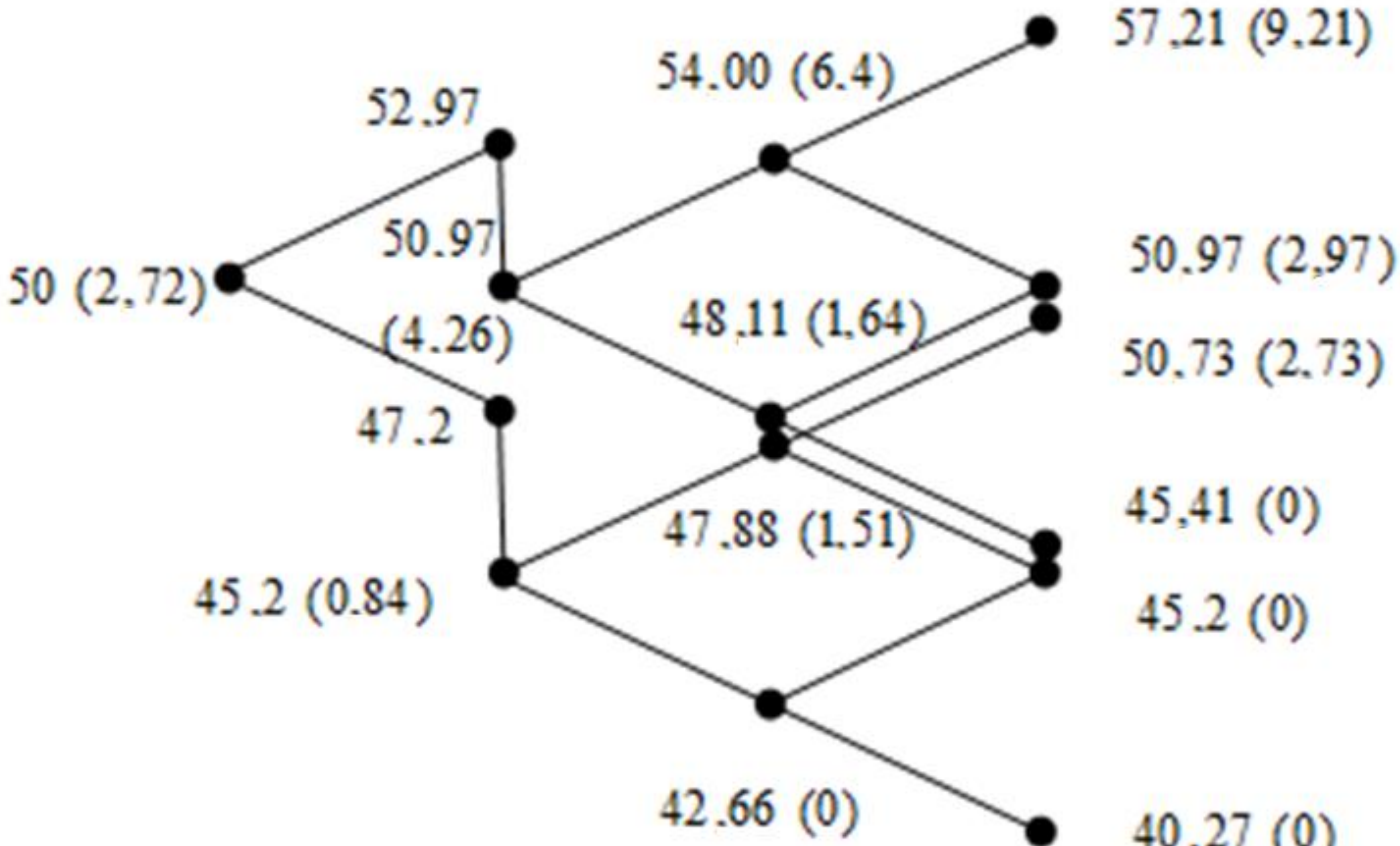
$$d = 1/u = 0.9439$$

$$1 - q = 0.442$$

$$e^{r\cdot\Delta t} = e^{0.1/12} = 1.008368$$

$$C = e^{-r\cdot\Delta t}\left(q\cdot C_u + (1-q)\cdot C_d\right)$$

$$q = \frac{e^{r\cdot\Delta t} - d}{u - d}$$


57.21 (9.21)
54.00 (6.4)
52.97
50.97
50 (2.72)
(4.26)
48.11 (1.64)
50.97 (2.97)
50.73 (2.73)
47.2
47.88 (1.51)
45.41 (0)
45.2 (0.84)
45.2 (0)
42.66 (0)
40.27 (0)

$D = 2, \quad K = 48$

Example - coherent binomial option tree (American put option)

$$S = 52 \quad K = 50 \quad D = 2.06 \quad r = 0.1 \quad \sigma = 0.4$$

$$\tau = 3.5 \qquad T = 5$$

$$u = e^{0.4 \cdot \sqrt{1/12}} = 1.122401$$

$$q = 0.507319$$

$$d = 1/u = 0.890947$$

$$1 - q = 0.492681$$

$$e^{r \cdot \Delta t} = e^{0.1/12} = 1.008368$$

$$P = e^{-r \cdot \Delta t}\left(q \cdot P_u + (1-q) \cdot P_d\right)$$

$$q = \frac{e^{r \cdot \Delta t} - d}{u - d}$$

S^*

$$D \cdot e^{-t \cdot \Delta t \cdot r} = 2.06 \cdot e^{-3.5 \cdot \frac{1}{12} \cdot 0.1} = 2.00078$$

49.99922	56.12	62.99	70.7	79.35	89.06
	44.55	49.999	56.12	62.99	70.7
		39.69	44.55	49.999	56.12
			35.36	39.69	44.55
				31.5	35.36
$+2.00078$	$+2.017527$	$+2.024938$	$+2.051435$		28.07
$+De^{-3.5 \cdot \frac{1}{12} \cdot 0.1}$	$+De^{-2.5 \cdot \frac{1}{12} \cdot 0.1}$	$+De^{-1.5 \cdot \frac{1}{12} \cdot 0.1}$	$+De^{-0.5 \cdot \frac{1}{12} \cdot 0.1}$		

S

52	58.14	65.01	72.75	79.35	89.06
	46.56	52.02	58.17	62.99	70.7
		41.71	46.598	49.999	56.12
			37.41	39.69	44.55
				31.5	35.36
					28.07

American put option

4.44	2.16	0.64	0	0	0
	6.86	3.77	1.3	0	0
		10.16	6.38	2.66	0
			14.22	10.31	5.45
				18.5	14.64
					21.93

Formula	K – S
0	– 29.35
0	–12.988
2.66	0
9.896	10.31
18.08	18.496

Option strategies

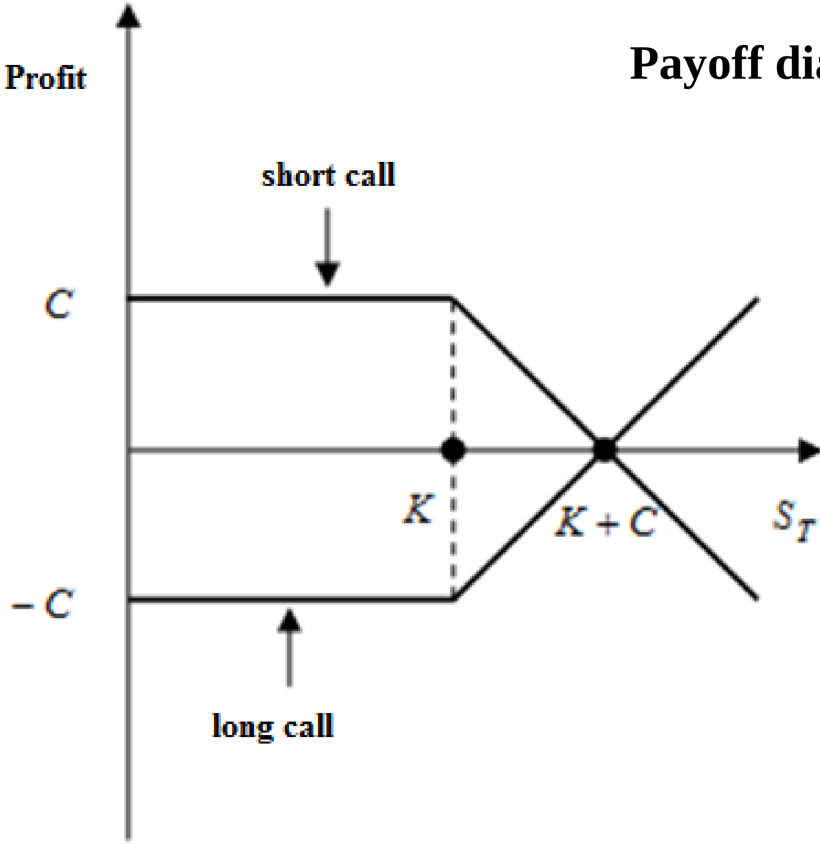
Uncovered

long call, short call, long put, short put

Covered

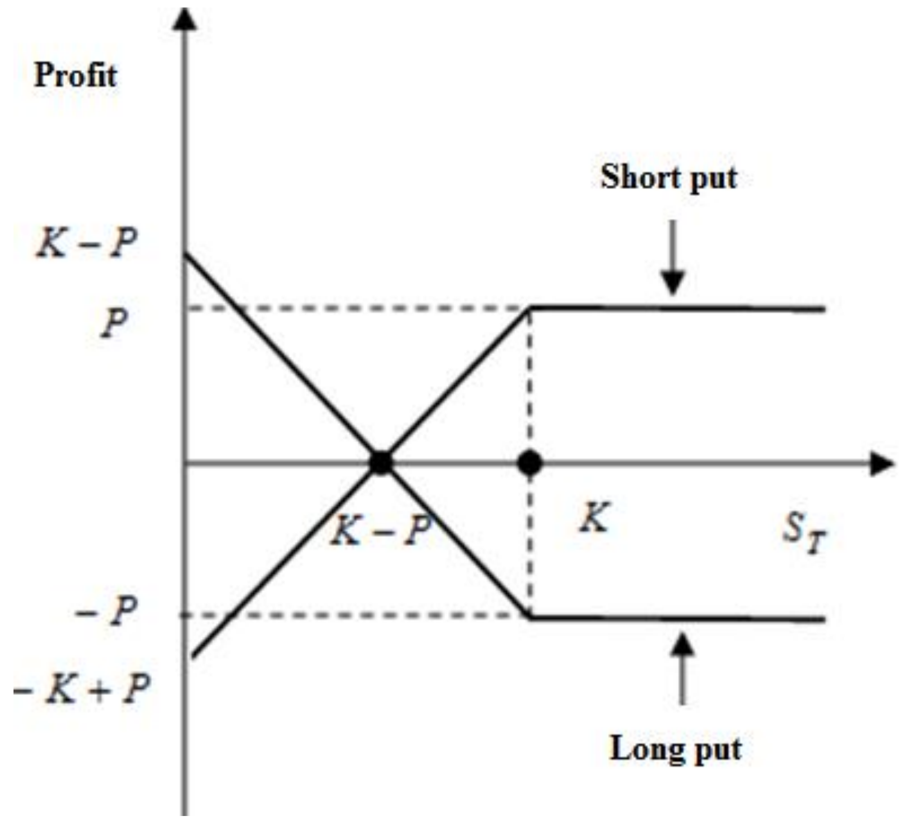
(option position that is offset by an equal and opposite position in underlying asset)

Payoff diagram



	Payoff	Profit
Long call	$\max\{S_T - K, 0\}$	$\max\{S_T - K, 0\} - C$
Short call	$\min\{K - S_T, 0\}$	$\min\{K - S_T, 0\} + C$

Payoff diagram



	Payoff	Profit
Long put	$\max\{K-S_T, 0\}$	$\max\{K-S_T, 0\}-P$
Short put	$\min\{S_T-K, 0\}$	$\min\{S_T-K, 0\}+P$

Covered option strategies

covered call – short call + long share

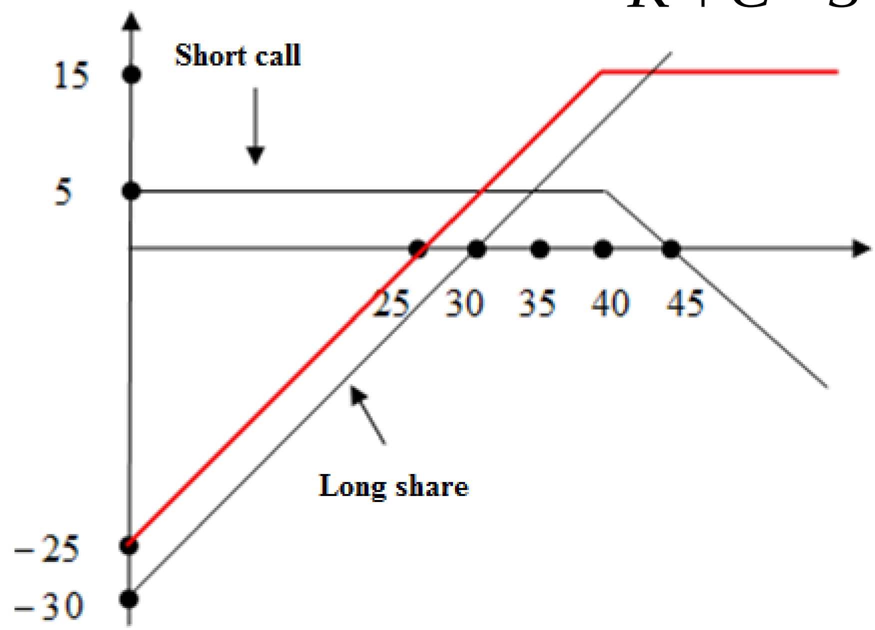
covered put – short put + short share

short call + long share

S_T	Short call	Long share $S_T - 30$	Profit
0	5	– 30	– 25
20	5	– 10	– 5
25	5	– 5	0
30	5	0	5
35	5	5	10
40	5	10	15
45	0	15	15
50	-5	20	15
55	-10	25	15
60	-15	30	15

$K = 40$ $C = 5$ $S = 30$

$K + C - S$

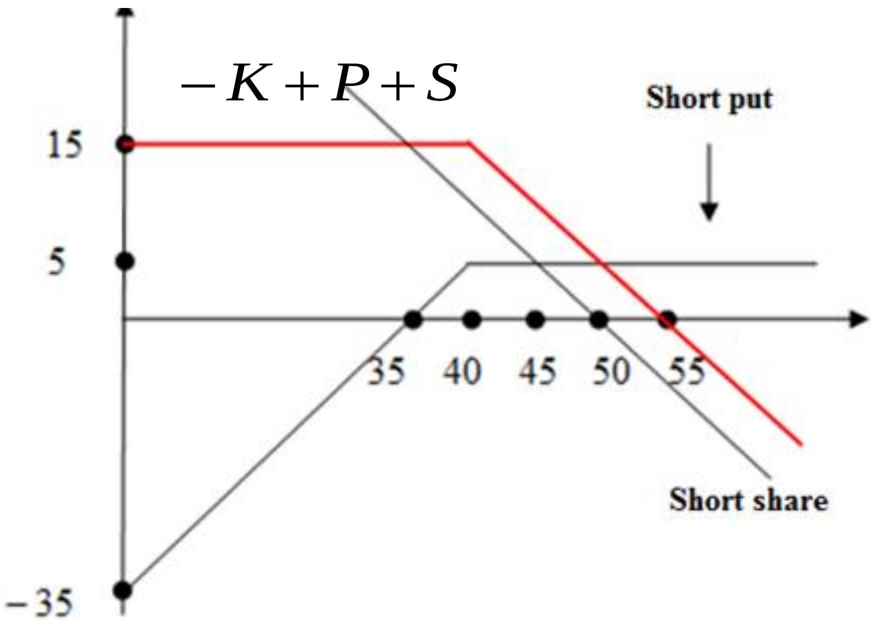


$$min\{K - S_T, 0\} + C$$

short put + short share

S_T	**Short put**	**Short share** $50 - S_T$	**Profit**
0	– 35	50	15
20	– 15	30	15
25	– 10	25	15
30	– 5	20	15
35	0	15	15
40	5	10	15
45	5	5	10
50	5	0	5
55	5	– 5	0
60	5	– 10	– 5

$K = 40 \quad P = 5 \quad S = 50$

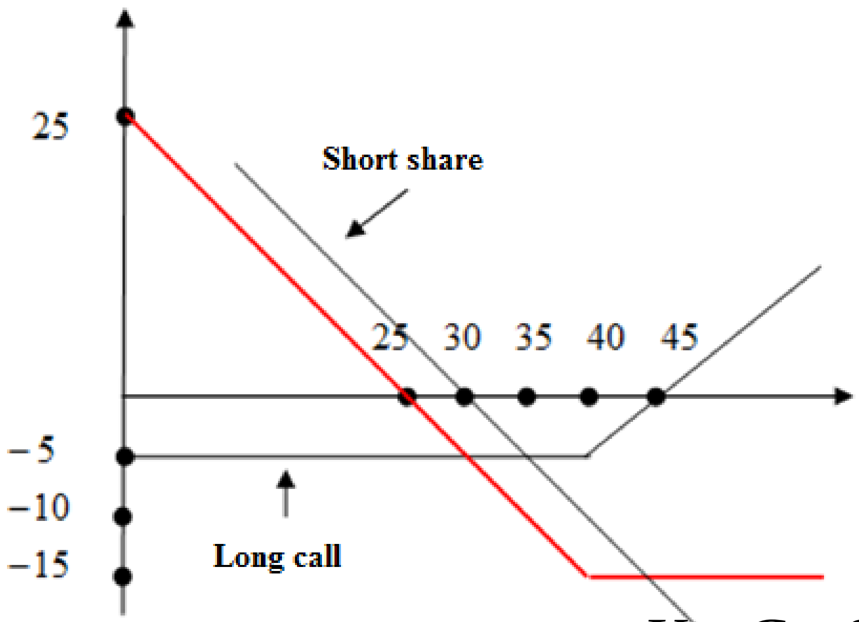


$min\{S_T - K, 0\} + P$

Long call + short share

S_T	Long call	Short share $30 - S_T$	Profit
0	– 5	30	25
20	– 5	10	5
25	– 5	5	0
30	– 5	0	–5
35	– 5	– 5	–10
40	– 5	– 10	–15
45	0	– 15	–15
50	5	– 20	–15
55	10	– 25	–15
60	15	– 30	–15

$K = 40$ $C = 5$ $S = 30$



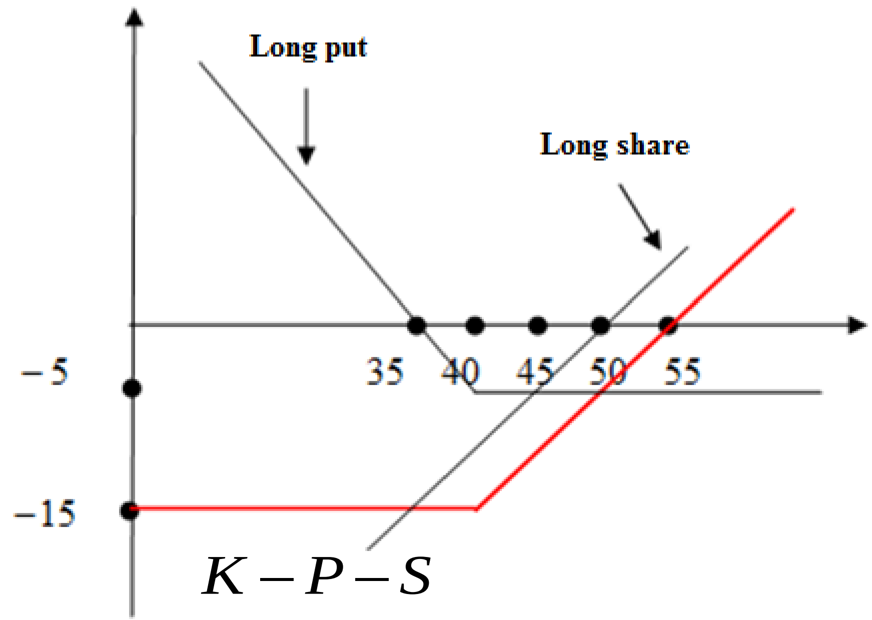
$-K - C + S$

$max\{S_T - K, 0\} - C$

Long put + long share

S_T	**Long put**	**Long share** $S_T - 50$	**Profit**
0	35	– 50	– 15
20	15	– 30	– 15
25	10	– 25	– 15
30	5	– 20	– 15
35	0	– 15	– 15
40	– 5	– 10	– 15
45	– 5	– 5	– 10
50	– 5	0	– 5
55	– 5	5	0
60	– 5	10	5

$K = 40 \qquad P = 5 \qquad S = 50$



$K - P - S$

$$max\{K - S_T, 0\} - P$$

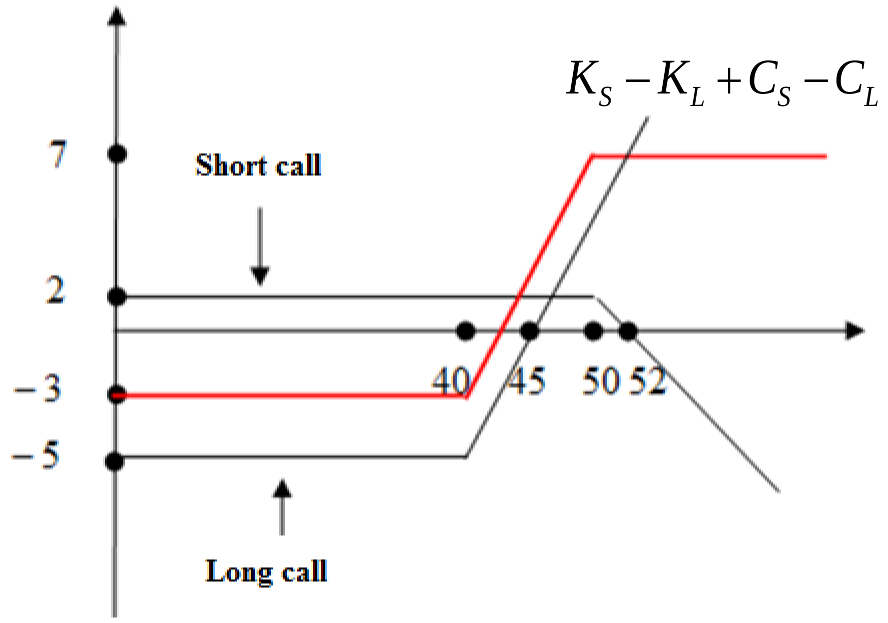
Bullish and bearish strategies

Bull call	$K_S > K_L \qquad C_S < C_L$
Bull put	$K_S > K_L \qquad P_S > P_L$
Bear call	$K_S < K_L \qquad C_S > C_L$
Bear put	$K_S < K_L \qquad P_S < P_L$

vertical bull call

S_T	Long call	Short call	Profit
20	– 5	2	– 3
25	– 5	2	– 3
30	– 5	2	– 3
35	– 5	2	– 3
40	– 5	2	–3
45	0	2	2
50	5	2	7
55	10	– 3	7
60	15	– 8	7
65	20	– 13	7

$K_L = 40 \quad C_L = 5 \quad K_S = 50 \quad C_S = 2$

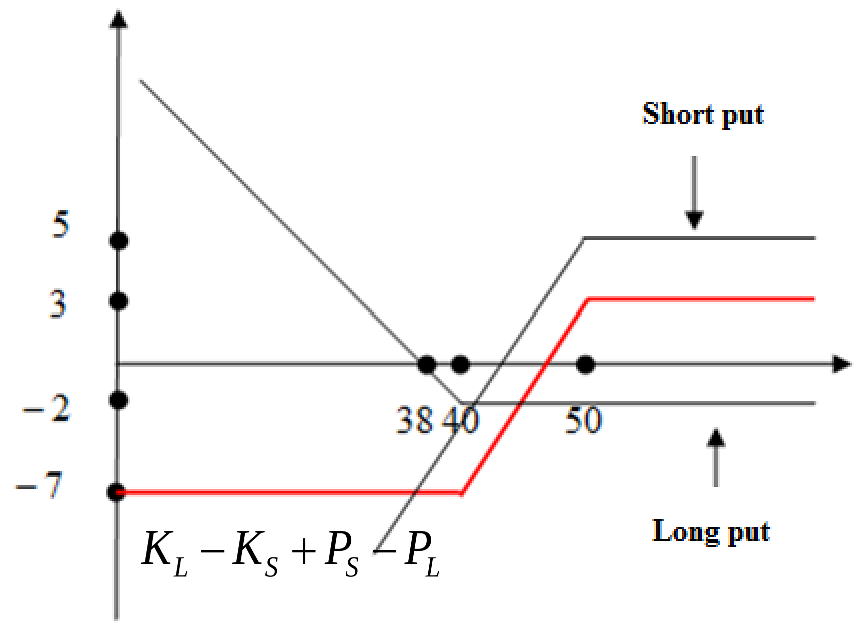


$max\{S_T - K_L, 0\} - C_L \qquad min\{K_S - S_T, 0\} + C_S$

vertical bull put

S_T	**Long put**	**Short put**	**Profit**
20	18	– 25	– 7
25	13	– 20	– 7
30	8	– 15	– 7
35	3	– 10	– 7
40	– 2	– 5	– 7
45	– 2	0	– 2
50	– 2	5	3
55	– 2	5	3
60	– 2	5	3
65	– 2	5	3

$K_L = 40 \quad P_L = 2 \quad K_S = 50 \quad P_S = 5$

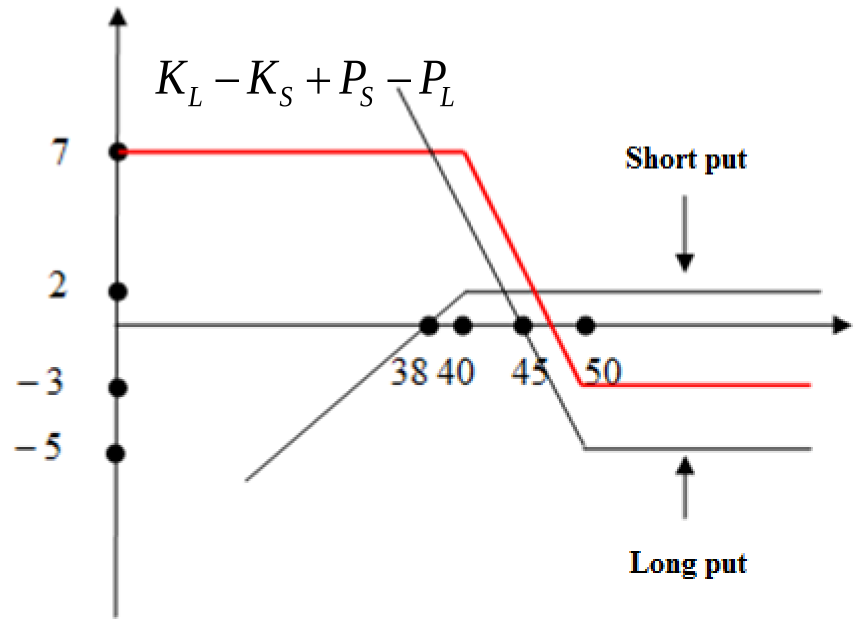


$$max\{K_L - S_T, 0\} - P_L \qquad min\{S_T - K_S, 0\} + P_S$$

vertical bear put

S_T	**Long put**	**Short put**	**Profit**
20	25	– 18	7
25	20	– 13	7
30	15	– 8	7
35	10	– 3	7
40	5	2	7
45	0	2	2
50	– 5	2	– 3
55	– 5	2	– 3
60	– 5	2	– 3
65	– 5	2	– 3

$K_L = 50 \quad P_L = 5 \quad K_S = 40 \quad P_S = 2$

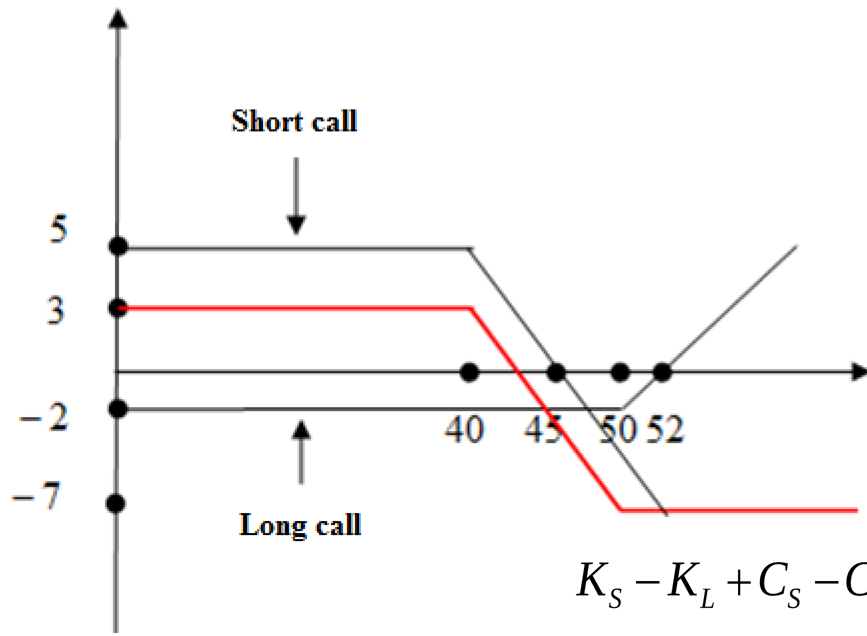


$max\{K_L - S_T, 0\} - P_L \qquad min\{S_T - K_S, 0\} + P_S$

vertical bear call

S_T	**Long call**	**Short call**	**Profit**
20	– 2	5	3
25	– 2	5	3
30	– 2	5	3
35	– 2	5	3
40	– 2	5	3
45	–2	0	– 2
50	– 2	– 5	– 7
55	3	– 10	– 7
60	8	– 15	– 7
65	13	– 20	– 7

$K_L = 50 \quad C_L = 2 \quad K_S = 40 \quad C_S = 5$



$K_S - K_L + C_S - C_L$

$max\{S_T - K_L, 0\} - C_L \qquad min\{K_S - S_T, 0\} + C_S$

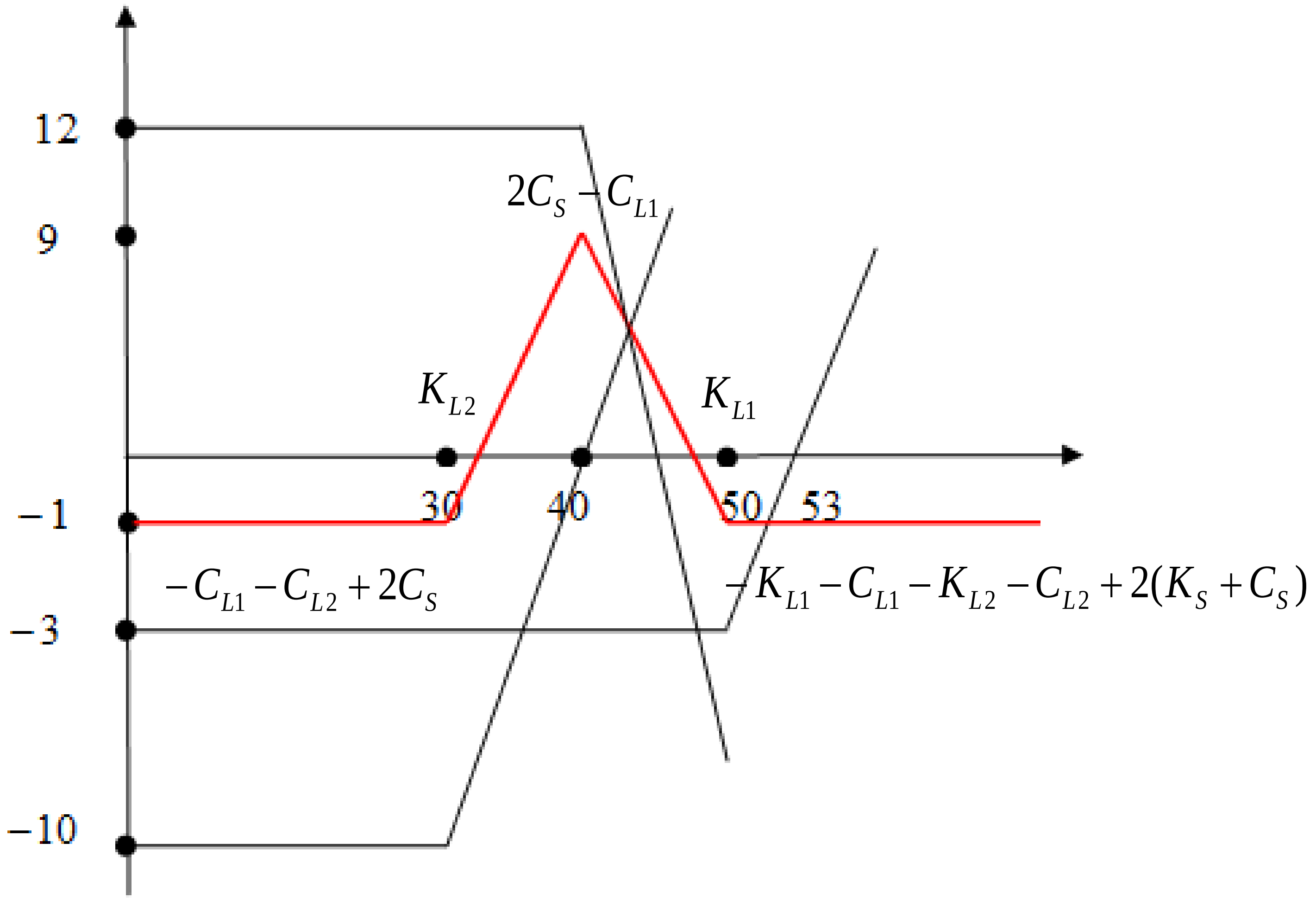
Call butterfly (long + short2+long)

$$\max\{S_T - K_L, 0\} - C_L \qquad \min\{K_S - S_T, 0\} + C_S$$

S_T	Long call 1	Short call x2	Long call 2	Profit
0	– 3	12=6*2	– 10	– 1
20	– 3	12	– 10	– 1
25	– 3	12	– 10	– 1
30	– 3	12	– 10	– 1
35	– 3	12	– 5	4
40	– 3	12	0	9
45	– 3	2=2*(40-45+6)	5	4
50	– 3	– 8	10	– 1
55	2	– 18	15	– 1
60	7	– 28	20	– 1

$K_{L1} = 50 \quad C_{L1} = 3 \qquad K_S = 40 \quad C_S = 6 \qquad K_{L2} = 30 \quad C_{L2} = 10$

Call butterfly



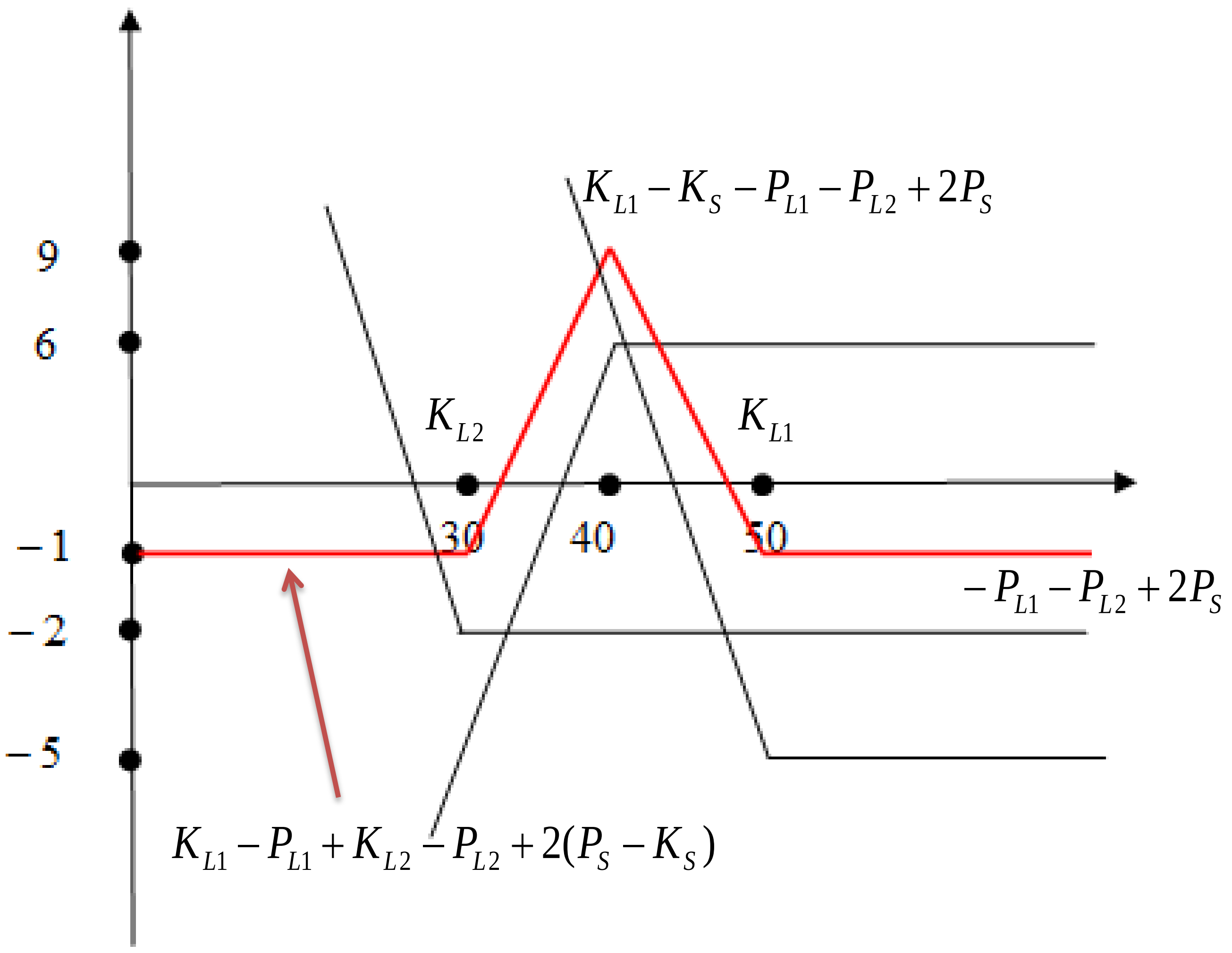
Put butterfly (long + short2+long)

$$max\{K_L - S_T, 0\} - P_L \qquad min\{S_T - K_S, 0\} + P_S$$

S_T	**Long put 1**	**Short put x2**	**Long put 2**	**Profit**
0	45	– 74	28	– 1
20	25	– 34	8	– 1
25	20	– 16	3	– 1
30	15	– 14	– 2	– 1
35	10	– 4	– 2	4
40	5	6	– 2	9
45	0	6	– 2	4
50	– 5	6	– 2	– 1
55	– 5	6	– 2	– 1
60	– 5	6	– 2	– 1

$K_{L1} = 50 \quad P_{L1} = 5 \qquad K_S = 40 \quad P_S = 3 \qquad K_{L2} = 30 \quad P_{L2} = 2$

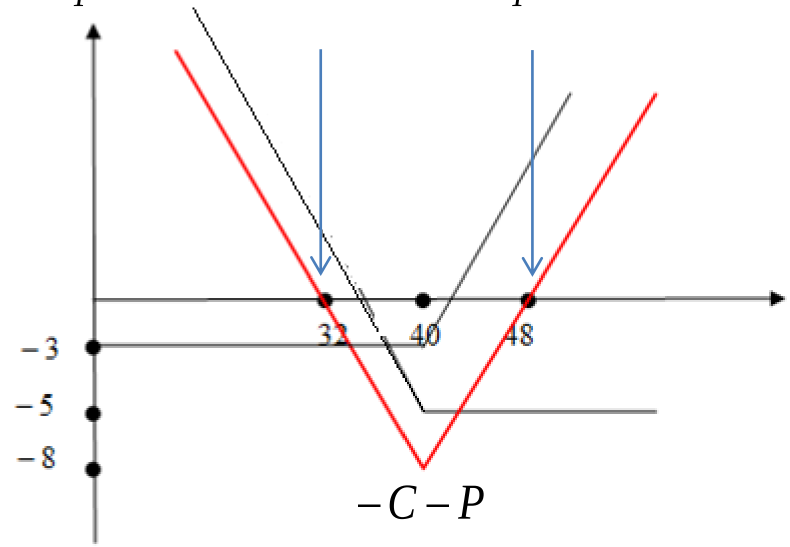
Put butterfly



Long straddle

S_T	**Long call**	**Long put**	**Profit**
0	– 3	35	32
20	– 3	15	12
25	– 3	10	7
30	– 3	5	2
35	– 3	0	– 3
40	– 3	– 5	– 8
45	2	– 5	– 3
50	7	– 5	2
55	12	– 5	7
60	17	– 5	12

$S_T = K - C - P$ $\quad$ $S_T = K + C + P$



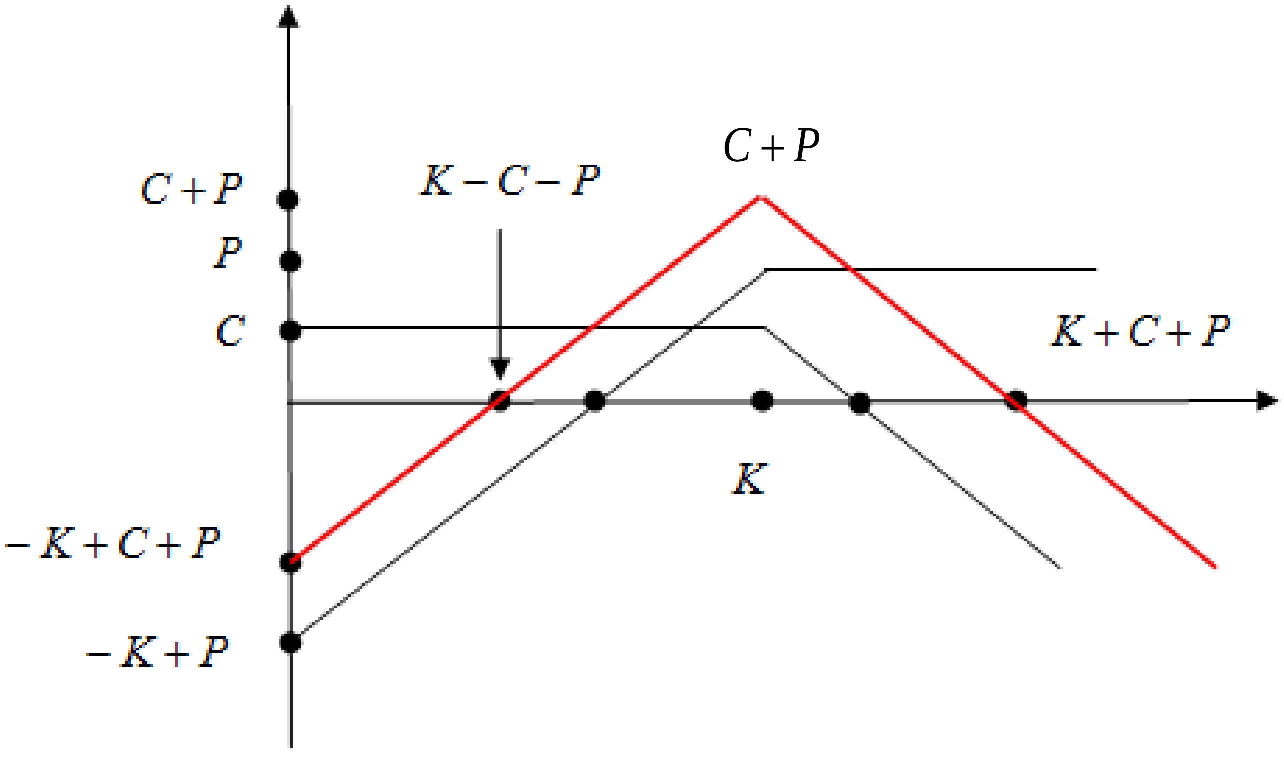
$-C-P$

$K = 40 \quad C = 3$

$K = 40 \quad P = 5$

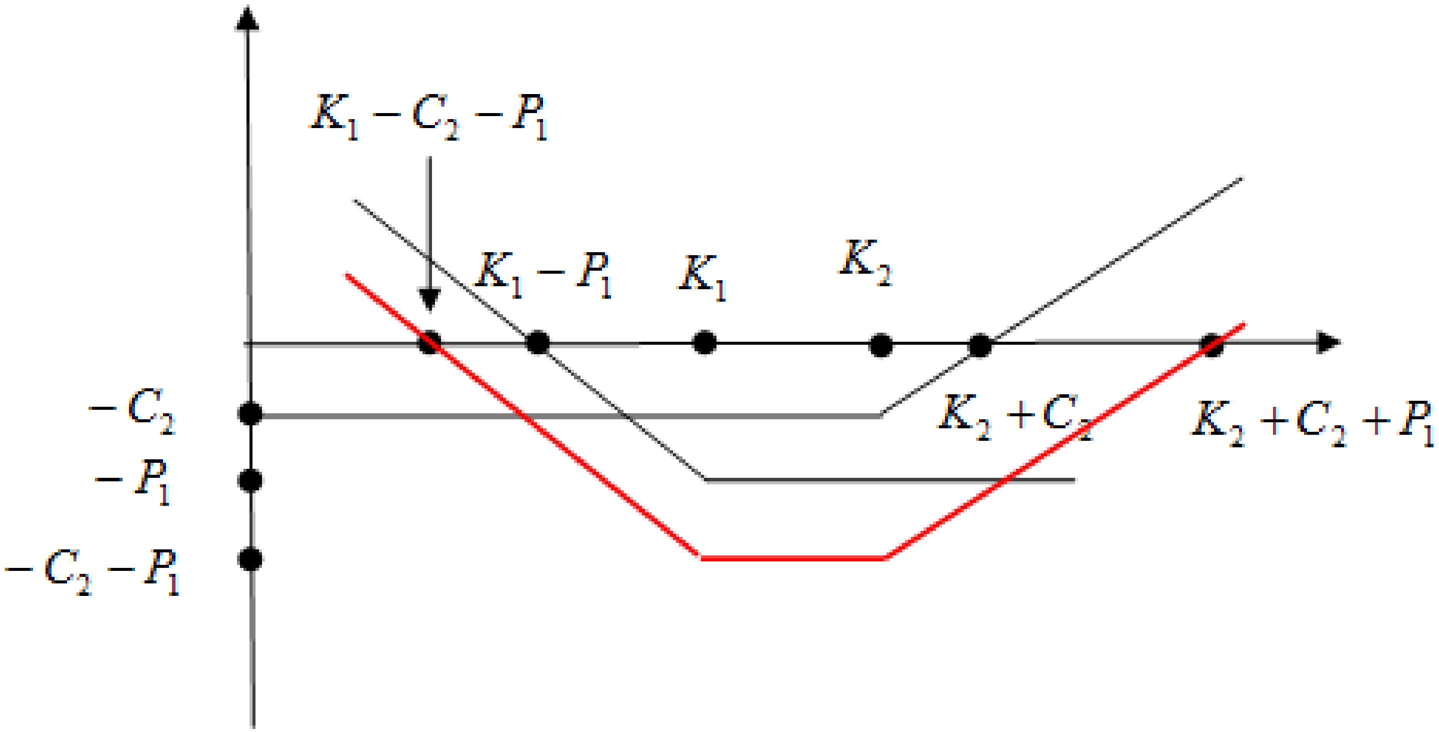
$\max\{S_T - K, 0\} - C \qquad \max\{K - S_T, 0\} - P$

Short straddle - short call + short put

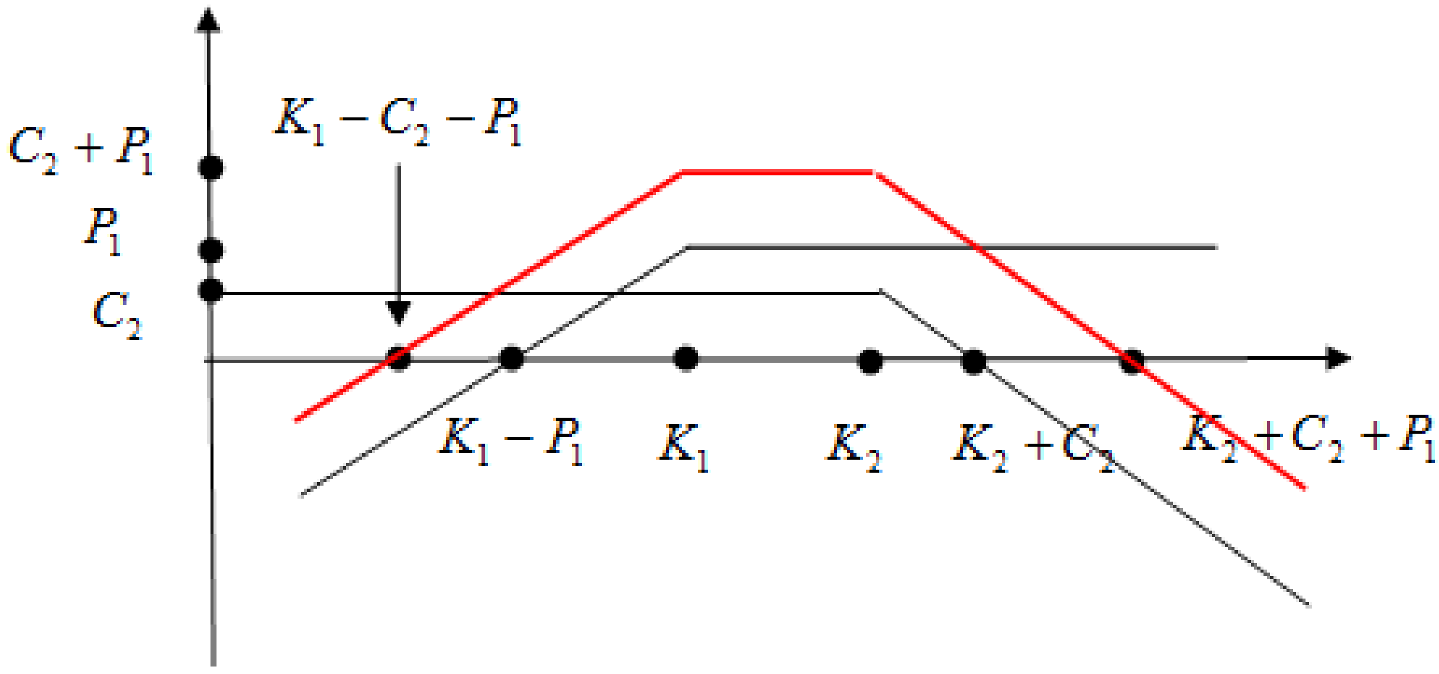


$$\min\{K-S_T,0\}+C \qquad \min\{S_T-K,0\}+P$$

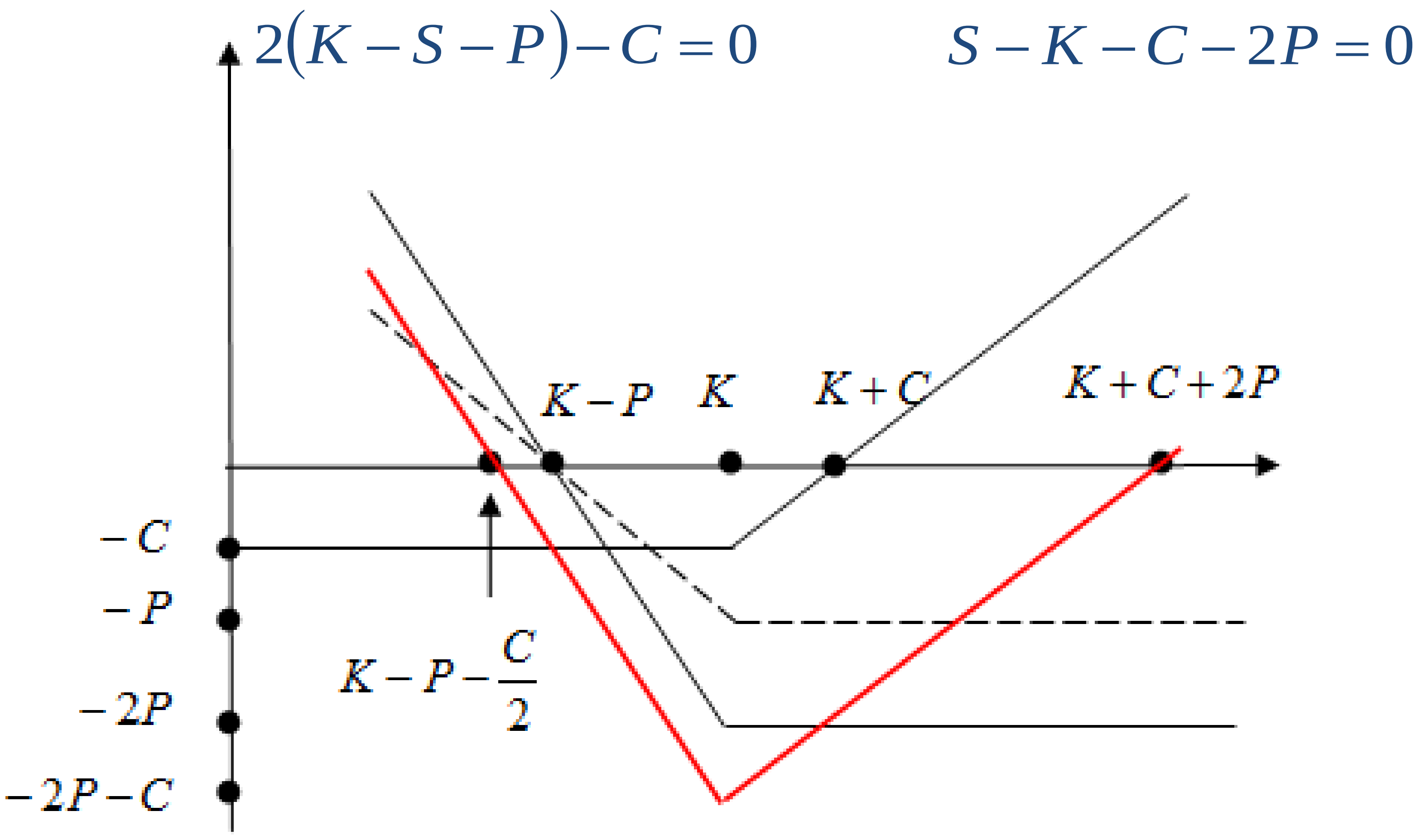
Long strangle - long call + long put



Short strangle - short call + short put



Strip



Strap

$$K - S - P - 2C = 0 \qquad 2(S - K - C) - P = 0$$