Financial Mathematics

Lecture 13-14

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Example

- Create a three-period binominal price tree and find the fair value of an European call and put options and an American put option on a nondividend-paying stock if the initial stock price is 62 PLN, the compound risk-free interest rate is 12% per annum, the stock volatility is 20%, the strike price of 60 PLN is expiring at the end of the third month (at the end of the third week).

Period - a month

$$u = e^{0.1\cdot\sqrt{1/12}} = 1.029288$$

$$d = 1/u = 0.971545$$

$$R = 1 + 0.12/12 = 1.01$$

$$q = 0.665965$$

$$1 - q = 0.334035$$

Period - a week

$$u = e^{0.1\cdot\sqrt{1/52}} = 1.013964$$

$$d = 1/u = 0.986228$$

$$R = 1 + 0.12/52 = 1.002308$$

$$q = 0.579736$$

$$1 - q = 0.420264$$

Three months

Price tree

62.00	63.82	65.68	67.61
	60.24	62.00	63.82
		58.52	60.24
			56.86

European call option

3.88	5.00	6.28	7.61
	1.76	2.59	3.82
		0.16	0.24
			0.00

European put option

0.11	0.00	0.00	0.00
	0.34	0.00	0.00
		1.04	0.00
			3.14

American put option

0.16	0.00	0.00	0.00
	0.49	0.00	0.00
		1.48	0.00
			3.14

Period - three weeks

Price tree

62.00	62.87	63.74	64.63
	61.15	62.00	62.87
		60.30	61.15
			59.47

European call option

2.45	3.14	3.88	4.63
	1.51	2.14	2.87
		0.66	1.15
			0.00

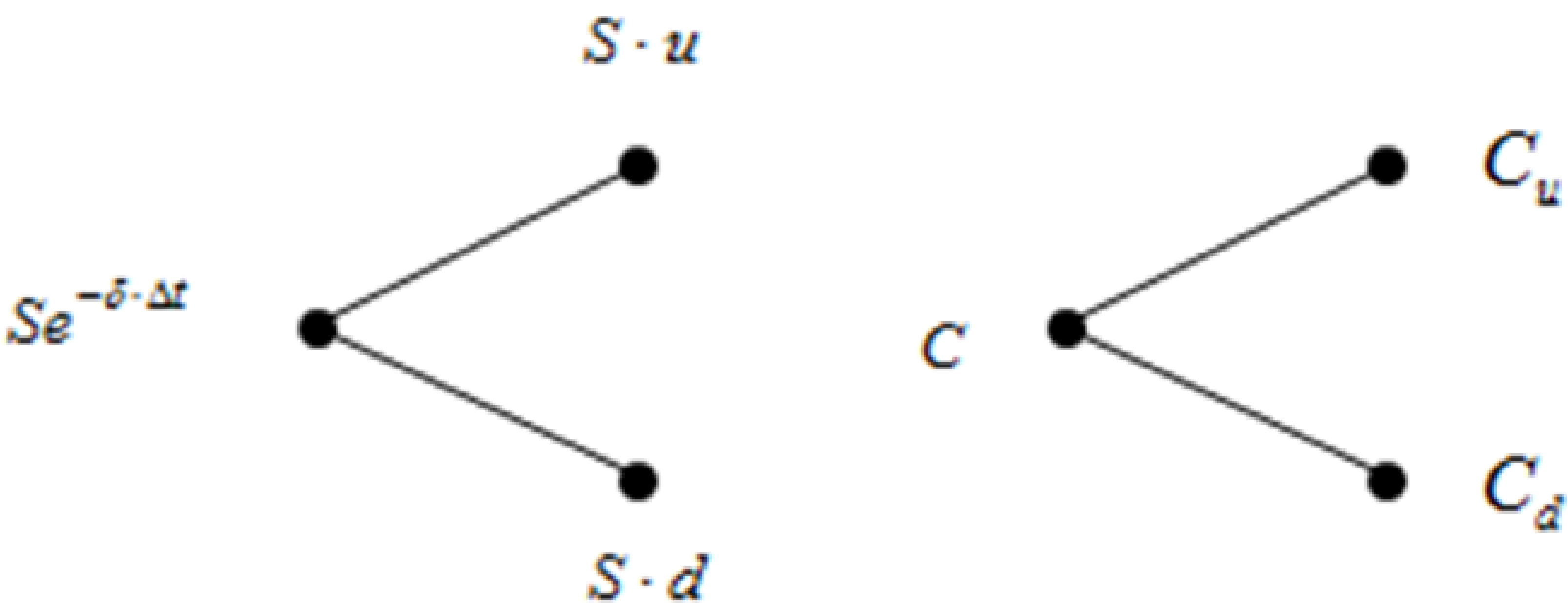
European put option

0.04	0.00	0.00	0.00
	0.09	0.00	0.00
		0.22	0.00
			0.53

American put option

0.04	0.00	0.00	0.00
	0.09	0.00	0.00
		0.22	0.00
			0.53

The underlying asset pays continuous dividend δ



$$u \cdot x + b \cdot e^{r \cdot \Delta t} = C_u$$

$$x = \frac{C_u - C_d}{u - d}$$

$$d \cdot x + b \cdot e^{r \cdot \Delta t} = C_d$$

$$b = e^{-r \cdot \Delta t}\left(\frac{u \cdot C_d - d \cdot C_u}{u - d}\right)$$

$$C = x \cdot e^{-\delta \cdot \Delta t} + b = \left(\frac{C_u - C_d}{u - d}\right)e^{-\delta \cdot \Delta t} + \left(\frac{u \cdot C_d - d \cdot C_u}{u - d}\right)e^{-r \cdot \Delta t}$$

$$C = x \cdot e^{-\delta \cdot \Delta t} + b = \left(\frac{C_u - C_d}{u - d} \right) e^{-\delta \cdot \Delta t} + \left(\frac{u \cdot C_d - d \cdot C_u}{u - d} \right) e^{-r \cdot \Delta t}$$

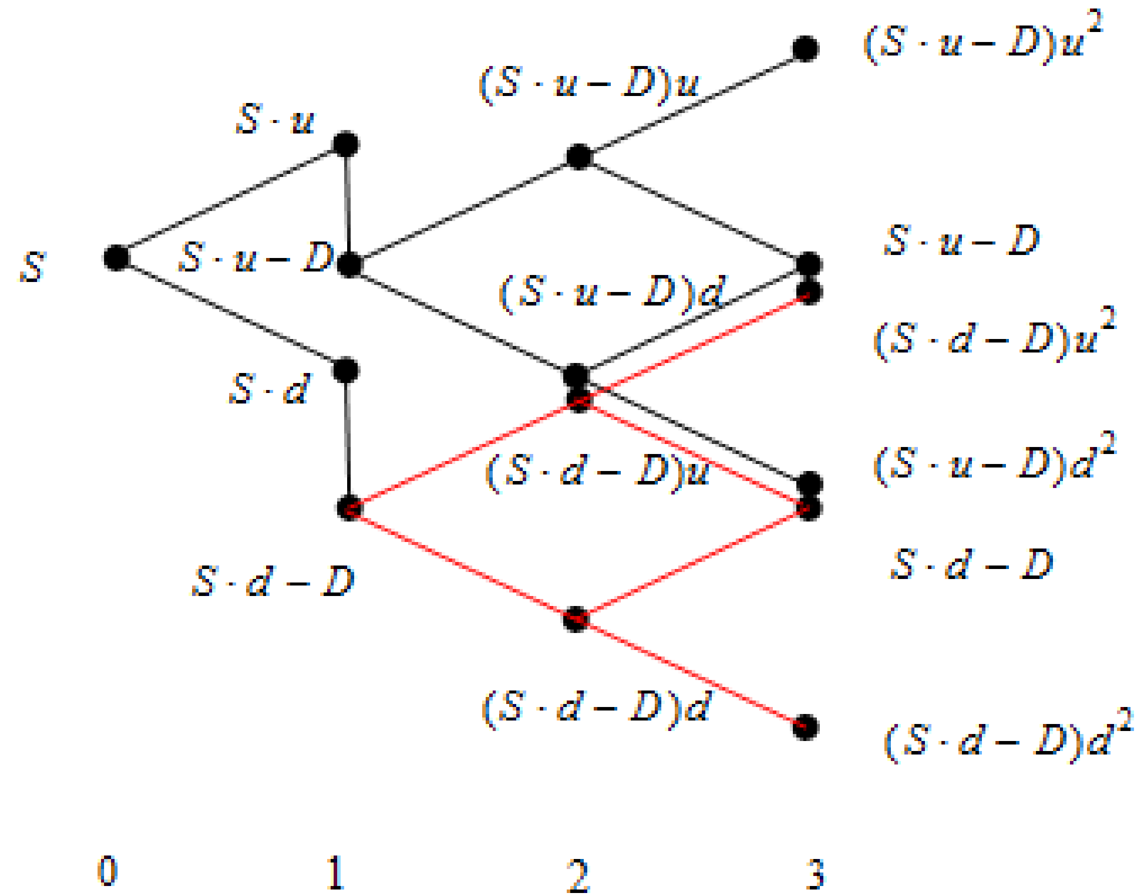
$$C = e^{-r \cdot \Delta t} \left[\left(\frac{C_u - C_d}{u - d} \right) e^{(r-\delta) \cdot \Delta t} + \left(\frac{u \cdot C_d - d \cdot C_u}{u - d} \right) \right]$$

$$C = e^{-r \cdot \Delta t} \left[\frac{e^{(r-\delta) \cdot \Delta t} - d}{u - d} C_u + \frac{u - e^{(r-\delta) \cdot \Delta t}}{u - d} C_d \right]$$

$$C = e^{-r \cdot \Delta t} \left(q \cdot C_u + (1 - q) \cdot C_d \right)$$

$$q = \frac{e^{(r-\delta) \cdot \Delta t} - d}{u - d}$$

Incoherent binomial option tree
(the underlying asset pays predictable income)



Example

- Find the fair value of an European call option using the incoherent binomial option tree if the underlying asset pays dividend of 2 PLN in half a month. The initial stock price is 50 PLN, the strike price of 48 PLN is expiring at the end of the third month, the continuously compounded risk-free interest rate is 10% per annum, and the stock volatility is 20%

$$u = e^{0.2\cdot\sqrt{1/12}} = 1.059434$$

$$q = 0.558$$

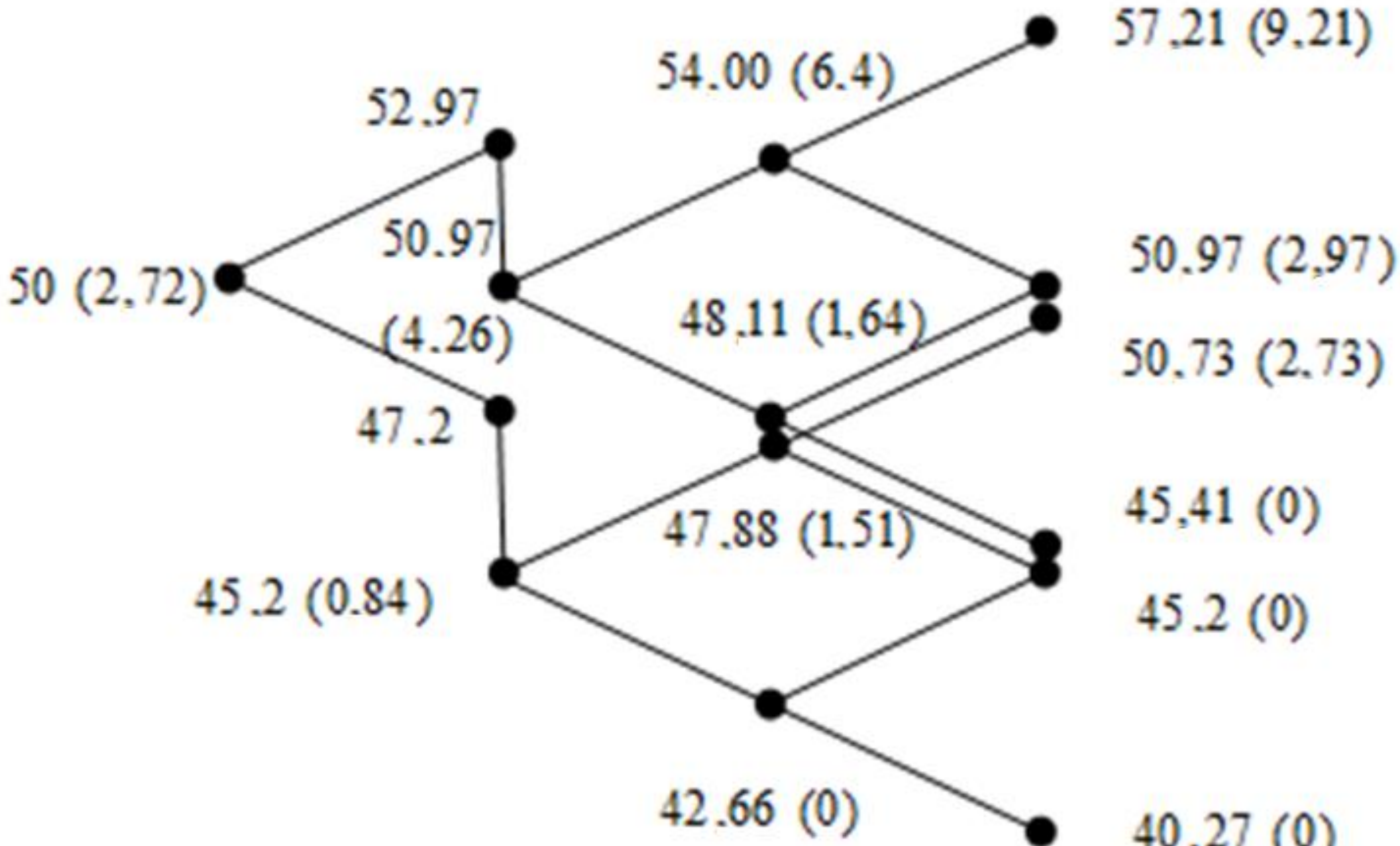
$$d = 1/u = 0.9439$$

$$1 - q = 0.442$$

$$e^{r\cdot\Delta t} = e^{0.1/12} = 1.008368$$

$$C = e^{-r\cdot\Delta t}\left(q\cdot C_u + (1-q)\cdot C_d\right)$$

$$q = \frac{e^{r\cdot\Delta t} - d}{u - d}$$


57,21 (9,21)
54.00 (6.4)
52,97
50,97
50 (2,72)
(4,26)
48,11 (1,64)
50,97 (2,97)
50,73 (2,73)
47,2
45,41 (0)
47,88 (1,51)
45,2 (0,84)
45,2 (0)
42,66 (0)
40,27 (0)

$D = 2, \quad K = 48$

Example - coherent binomial option tree (American put option)

$$S = 52 \quad K = 50 \quad D = 2.06 \quad r = 0.1 \quad \sigma = 0.4$$

$$\tau = 3.5 \qquad T = 5$$

$$u = e^{0.4 \cdot \sqrt{1/12}} = 1.122401$$

$$q = 0.507319$$

$$d = 1/u = 0.890947$$

$$1 - q = 0.492681$$

$$e^{r \cdot \Delta t} = e^{0.1/12} = 1.008368$$

$$P = e^{-r \cdot \Delta t}\left(q \cdot P_u + (1-q) \cdot P_d\right)$$

$$q = \frac{e^{r \cdot \Delta t} - d}{u - d}$$

$$S^*$$

$$D \cdot e^{-t \cdot \Delta t \cdot r} = 2.06 \cdot e^{-3.5 \cdot \frac{1}{12} \cdot 0.1} = 2.00078$$

49.99922	56.12	62.99	70.7	79.35	89.06
	44.55	49.999	56.12	62.99	70.7
		39.69	44.55	49.999	56.12
			35.36	39.69	44.55
				31.5	35.36
					28.07
$+2.00078$	$+2.017527$	$+2.024938$	$+2.051435$		
$+De^{-3.5 \cdot \frac{1}{12} \cdot 0.1}$	$+De^{-2.5 \cdot \frac{1}{12} \cdot 0.1}$	$+De^{-1.5 \cdot \frac{1}{12} \cdot 0.1}$	$+De^{-0.5 \cdot \frac{1}{12} \cdot 0.1}$		

S

52	58.14	65.01	72.75	79.35	89.06
	46.56	52.02	58.17	62.99	70.7
		41.71	46.598	49.999	56.12
			37.41	39.69	44.55
				31.5	35.36
					28.07

American put option

4.44	2.16	0.64	0	0	0
	6.86	3.77	1.3	0	0
		10.16	6.38	2.66	0
			14.22	10.31	5.45
				18.5	14.64
					21.93

Formula	K – S
0	– 29.35
0	–12.988
2.66	0
9.896	10.31
18.08	18.496

Option strategies

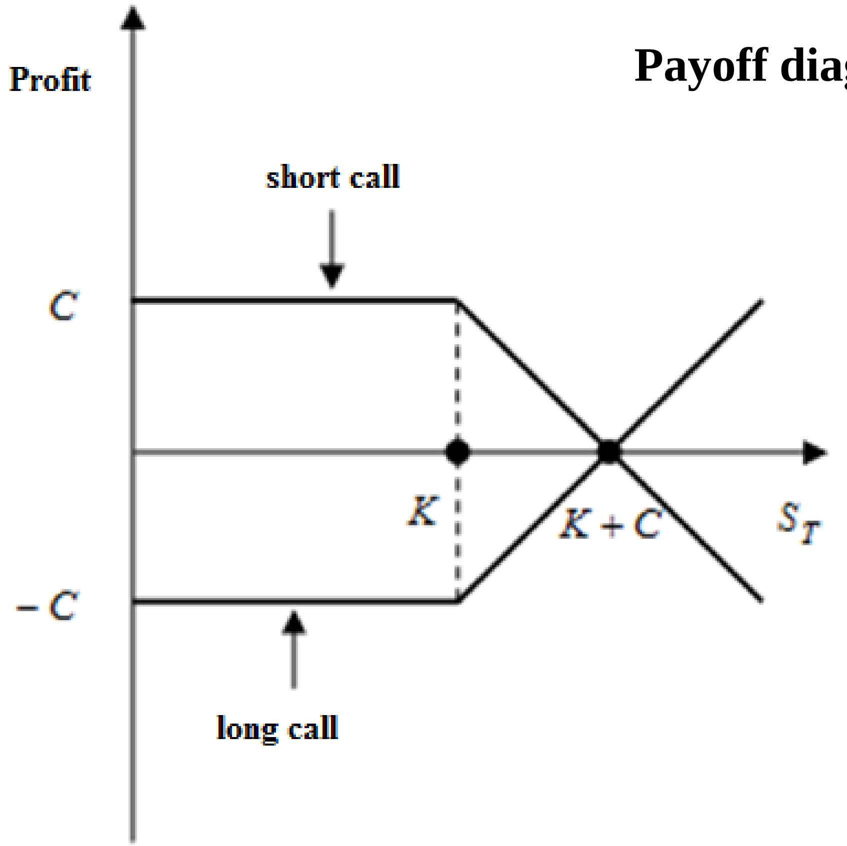
Uncovered

long call, short call, long put, short put

Covered

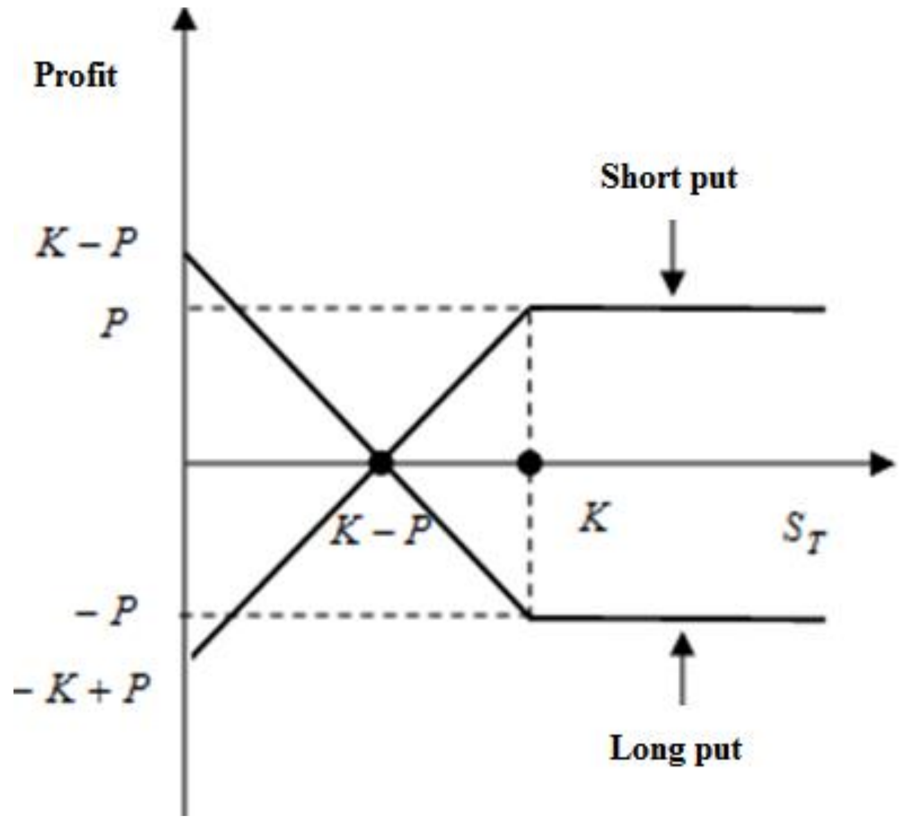
(option position that is offset by an equal and opposite position in underlying asset)

Payoff diagram



	Payoff	Profit
Long call	$\max\{S_T - K, 0\}$	$\max\{S_T - K, 0\} - C$
Short call	$\min\{K - S_T, 0\}$	$\min\{K - S_T, 0\} + C$

Payoff diagram



	Payoff	Profit
Long put	$\max\{K - S_T, 0\}$	$\max\{K - S_T, 0\} - P$
Short put	$\min\{S_T - K, 0\}$	$\min\{S_T - K, 0\} + P$

Covered option strategies

covered call – short call + long share

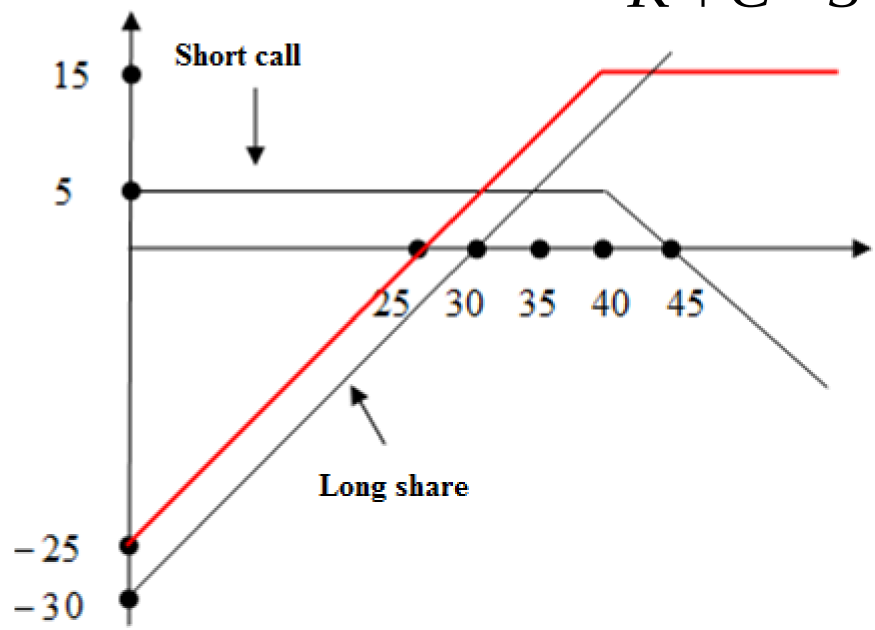
covered put – short put + short share

short call + long share

S_T	Short call	Long share $S_T - 30$	Profit
0	5	– 30	– 25
20	5	– 10	– 5
25	5	– 5	0
30	5	0	5
35	5	5	10
40	5	10	15
45	0	15	15
50	-5	20	15
55	-10	25	15
60	-15	30	15

$K = 40$ $C = 5$ $S = 30$

$K + C - S$

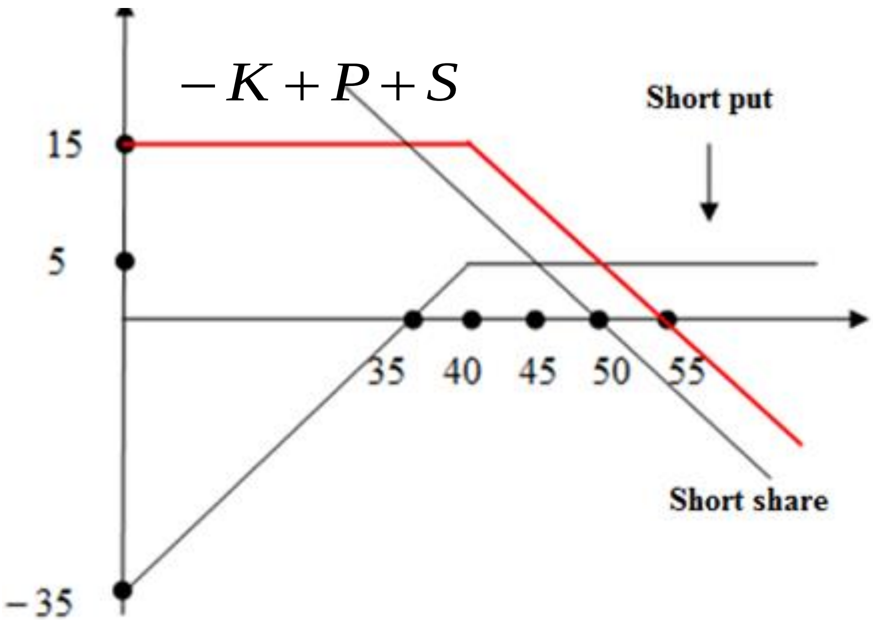


$$min\{K - S_T, 0\} + C$$

short put + short share

S_T	**Short put**	**Short share** $50 - S_T$	**Profit**
0	– 35	50	15
20	– 15	30	15
25	– 10	25	15
30	– 5	20	15
35	0	15	15
40	5	10	15
45	5	5	10
50	5	0	5
55	5	– 5	0
60	5	– 10	– 5

$K = 40 \quad P = 5 \quad S = 50$

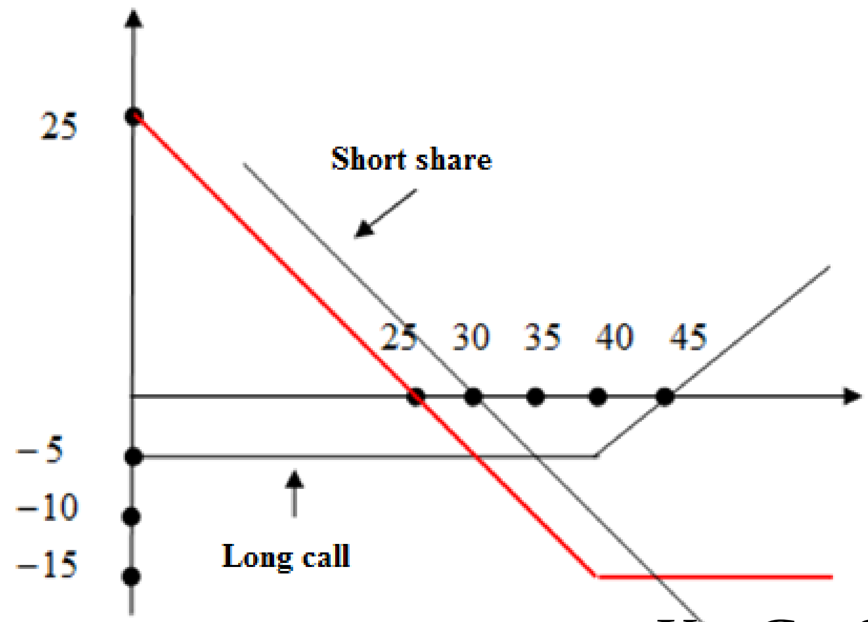


$$min\{S_T - K, 0\} + P$$

Long call + short share

S_T	**Long call**	**Short share** $30-S_T$	**Profit**
0	– 5	30	25
20	– 5	10	5
25	– 5	5	0
30	– 5	0	–5
35	– 5	– 5	–10
40	– 5	– 10	–15
45	0	– 15	–15
50	5	– 20	–15
55	10	– 25	–15
60	15	– 30	–15

$K=40 \qquad C=5 \qquad S=30$



$-K-C+S$

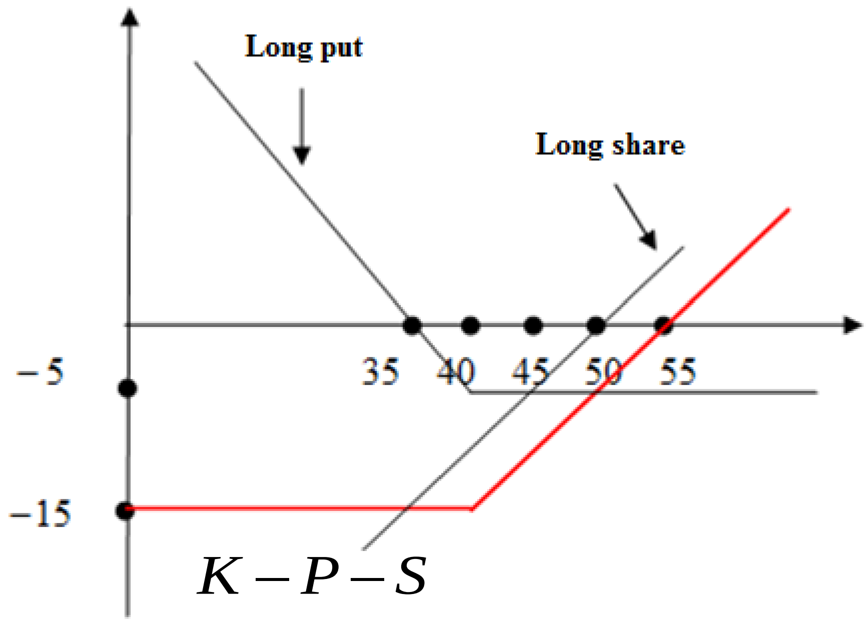
$max\{S_T - K, 0\} - C$

Long put + long share

S_T	**Long put**	**Long share** $S_T - 50$	**Profit**
0	35	– 50	– 15
20	15	– 30	– 15
25	10	– 25	– 15
30	5	– 20	– 15
35	0	– 15	– 15
40	– 5	– 10	– 15
45	– 5	– 5	– 10
50	– 5	0	– 5
55	– 5	5	0
60	– 5	10	5

$$max\{K - S_T, 0\} - P$$

$K = 40 \quad P = 5 \quad S = 50$



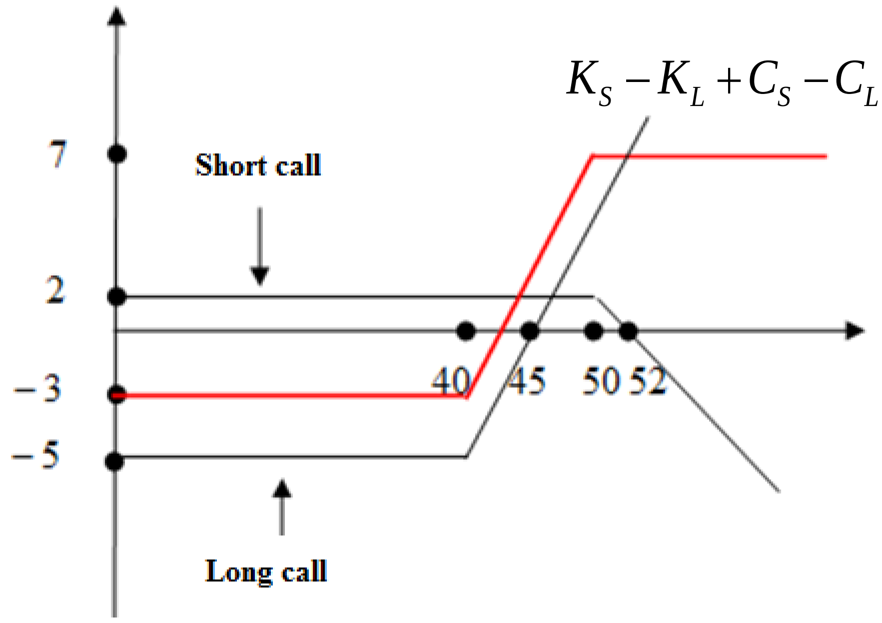
Bullish and bearish strategies

Bull call	$K_S > K_L$	$C_S < C_L$
Bull put	$K_S > K_L$	$P_S > P_L$
Bear call	$K_S < K_L$	$C_S > C_L$
Bear put	$K_S < K_L$	$P_S < P_L$

vertical bull call

S_T	Long call	Short call	Profit
20	– 5	2	– 3
25	– 5	2	– 3
30	– 5	2	– 3
35	– 5	2	– 3
40	– 5	2	–3
45	0	2	2
50	5	2	7
55	10	– 3	7
60	15	– 8	7
65	20	– 13	7

$K_L = 40 \quad C_L = 5 \quad K_S = 50 \quad C_S = 2$

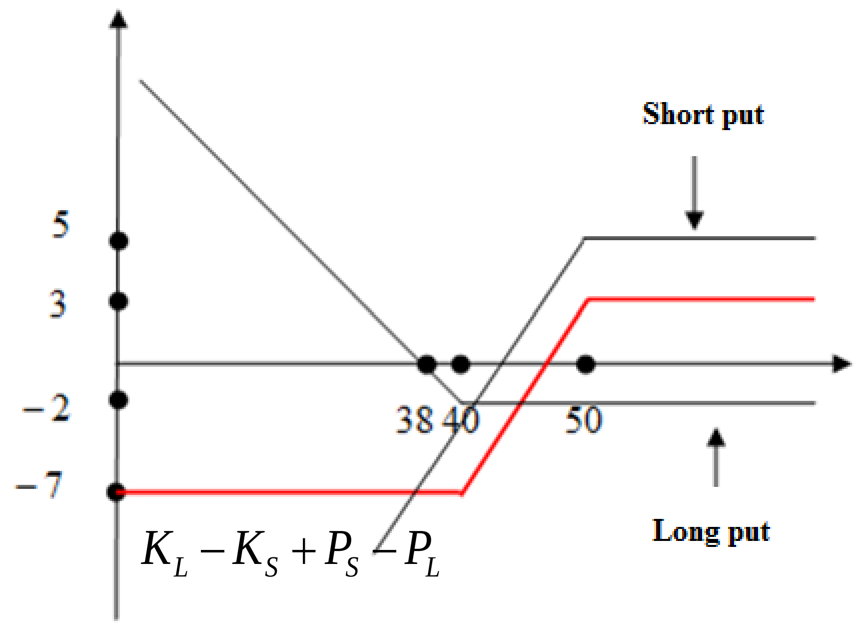


$max\{S_T - K_L, 0\} - C_L \qquad min\{K_S - S_T, 0\} + C_S$

vertical bull put

S_T	**Long put**	**Short put**	**Profit**
20	18	– 25	– 7
25	13	– 20	– 7
30	8	– 15	– 7
35	3	– 10	– 7
40	– 2	– 5	– 7
45	– 2	0	– 2
50	– 2	5	3
55	– 2	5	3
60	– 2	5	3
65	– 2	5	3

$K_L = 40 \quad P_L = 2 \quad K_S = 50 \quad P_S = 5$

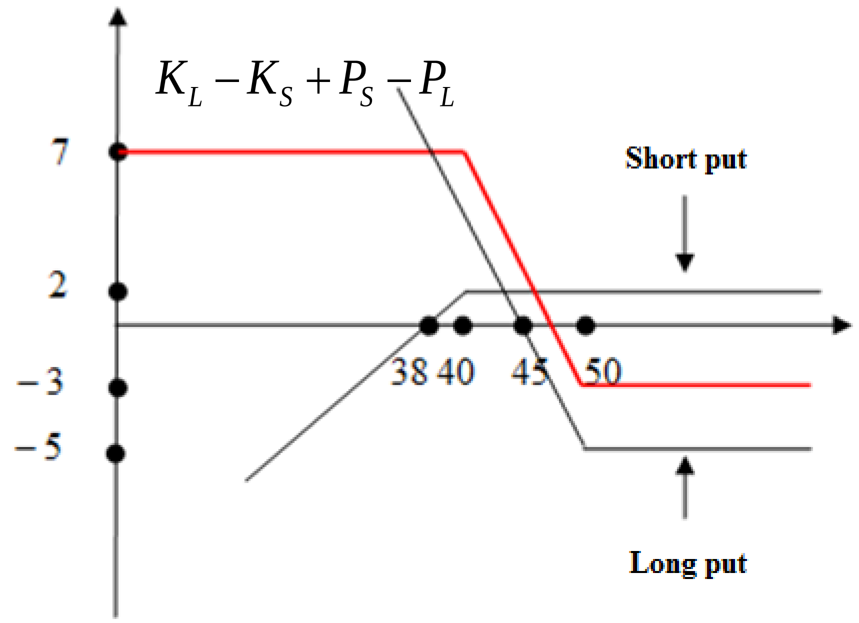


$$max\{K_L - S_T, 0\} - P_L \qquad min\{S_T - K_S, 0\} + P_S$$

vertical bear put

S_T	**Long put**	**Short put**	**Profit**
20	25	– 18	7
25	20	– 13	7
30	15	– 8	7
35	10	– 3	7
40	5	2	7
45	0	2	2
50	– 5	2	– 3
55	– 5	2	– 3
60	– 5	2	– 3
65	– 5	2	– 3

$K_L = 50 \quad P_L = 5 \quad K_S = 40 \quad P_S = 2$

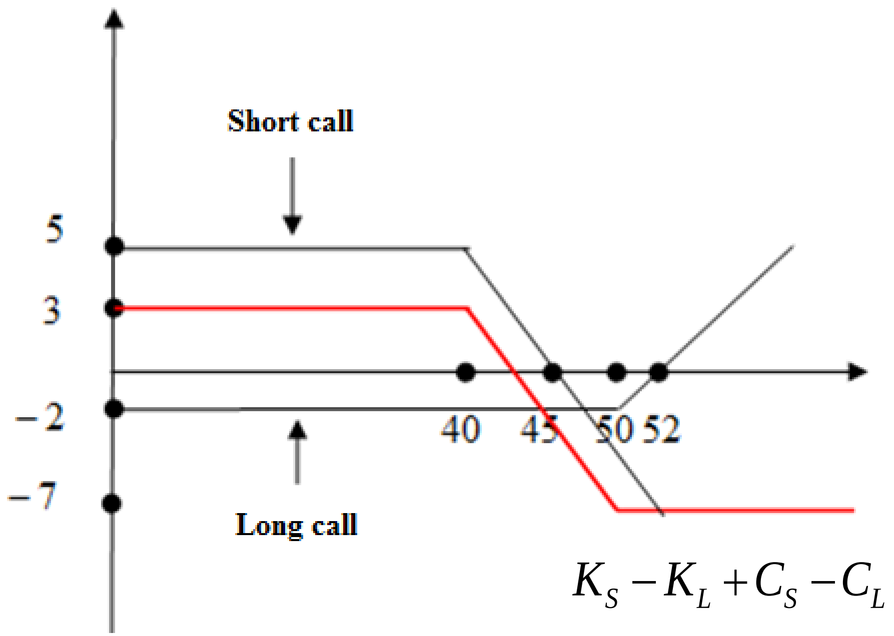


$max\{K_L - S_T, 0\} - P_L \qquad min\{S_T - K_S, 0\} + P_S$

vertical bear call

S_T	**Long call**	**Short call**	**Profit**
20	– 2	5	3
25	– 2	5	3
30	– 2	5	3
35	– 2	5	3
40	– 2	5	3
45	–2	0	– 2
50	– 2	– 5	– 7
55	3	– 10	– 7
60	8	– 15	– 7
65	13	– 20	– 7

$K_L = 50 \quad C_L = 2 \quad K_S = 40 \quad C_S = 5$



$K_S - K_L + C_S - C_L$

$max\{S_T - K_L, 0\} - C_L \qquad min\{K_S - S_T, 0\} + C_S$

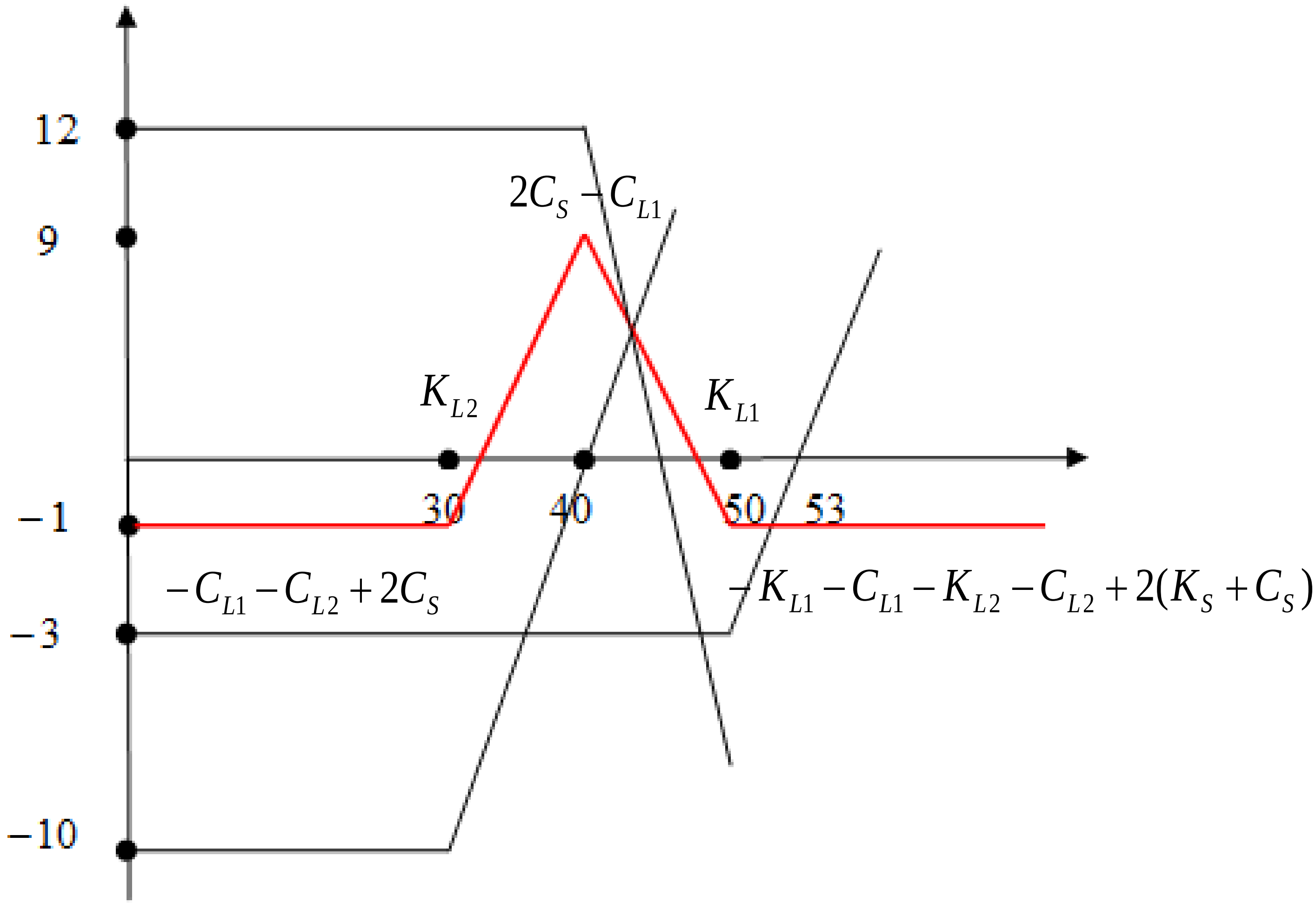
Call butterfly (long + short2+long)

$$max\{S_T - K_L, 0\} - C_L \qquad min\{K_S - S_T, 0\} + C_S$$

S_T	Long call 1	Short call x2	Long call 2	Profit
0	– 3	12=6*2	– 10	– 1
20	– 3	12	– 10	– 1
25	– 3	12	– 10	– 1
30	– 3	12	– 10	– 1
35	– 3	12	– 5	4
40	– 3	12	0	9
45	– 3	2=2*(40-45+6)	5	4
50	– 3	– 8	10	– 1
55	2	– 18	15	– 1
60	7	– 28	20	– 1

$K_{L1} = 50 \quad C_{L1} = 3 \qquad K_S = 40 \quad C_S = 6 \qquad K_{L2} = 30 \quad C_{L2} = 10$

Call butterfly



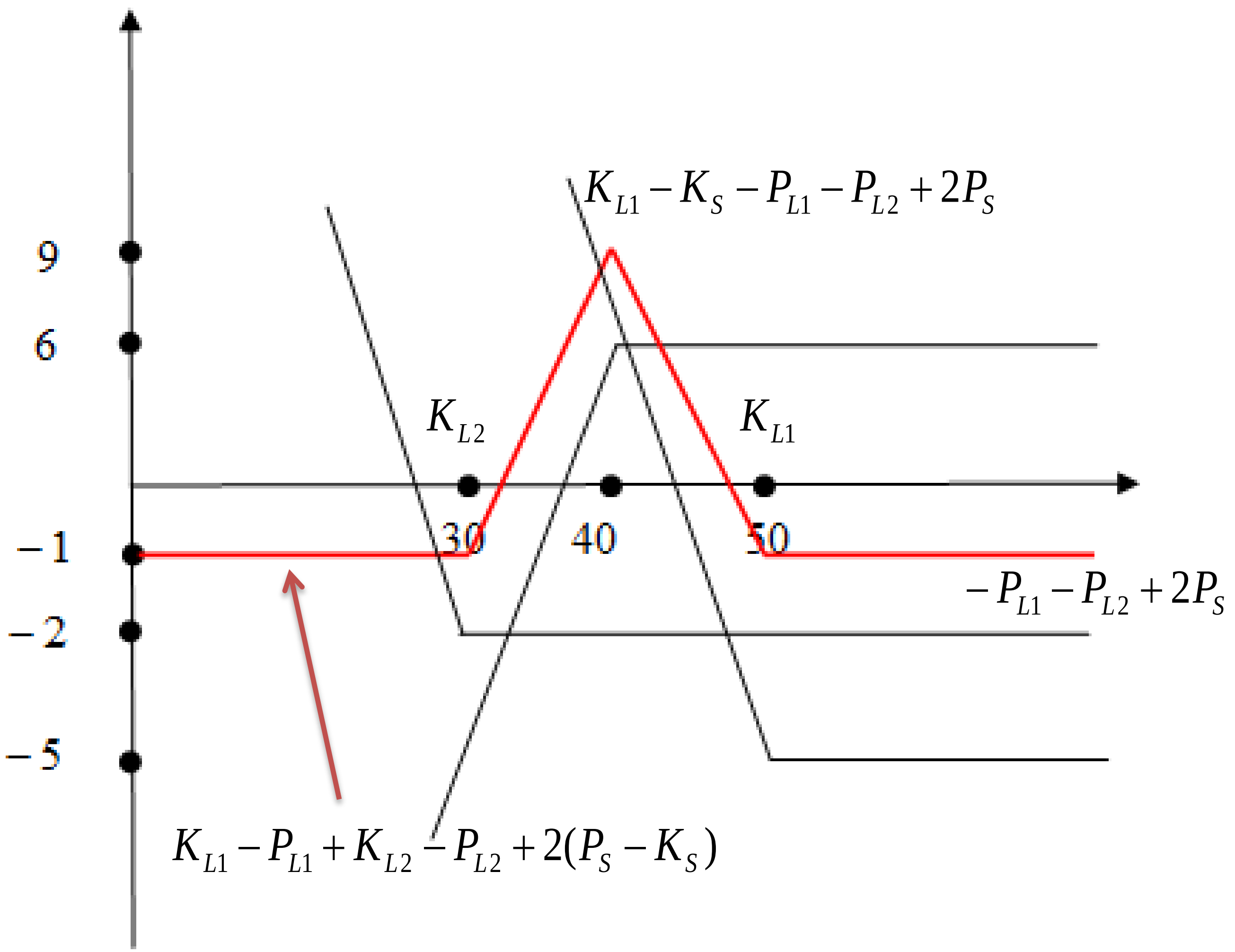
Put butterfly (long + short2+long)

$$max\{K_L - S_T, 0\} - P_L \qquad min\{S_T - K_S, 0\} + P_S$$

S_T	Long put 1	Short put x2	Long put 2	Profit
0	45	– 74	28	– 1
20	25	– 34	8	– 1
25	20	– 16	3	– 1
30	15	– 14	– 2	– 1
35	10	– 4	– 2	4
40	5	6	– 2	9
45	0	6	– 2	4
50	– 5	6	– 2	– 1
55	– 5	6	– 2	– 1
60	– 5	6	– 2	– 1

$K_{L1} = 50 \quad P_{L1} = 5 \qquad K_S = 40 \quad P_S = 3 \qquad K_{L2} = 30 \quad P_{L2} = 2$

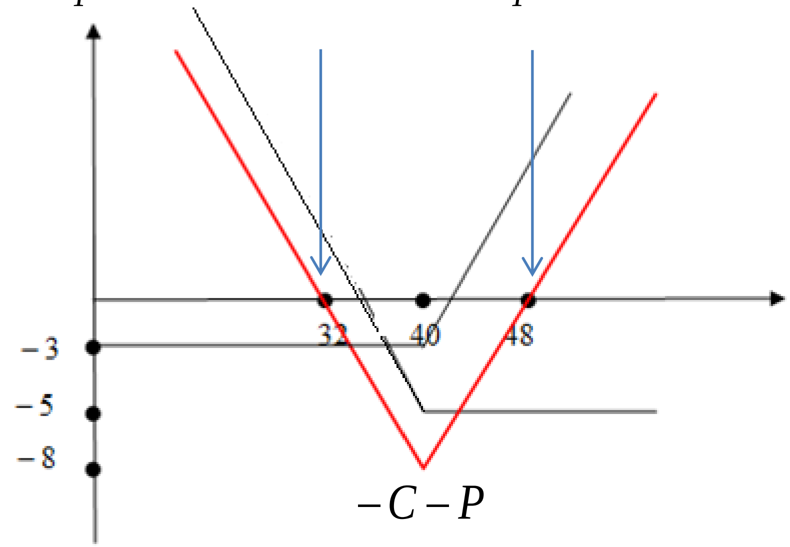
Put butterfly



Long straddle

S_T	Long call	Long put	Profit
0	– 3	35	32
20	– 3	15	12
25	– 3	10	7
30	– 3	5	2
35	– 3	0	– 3
40	– 3	– 5	– 8
45	2	– 5	– 3
50	7	– 5	2
55	12	– 5	7
60	17	– 5	12

$S_T = K - C - P$ $\quad$ $S_T = K + C + P$



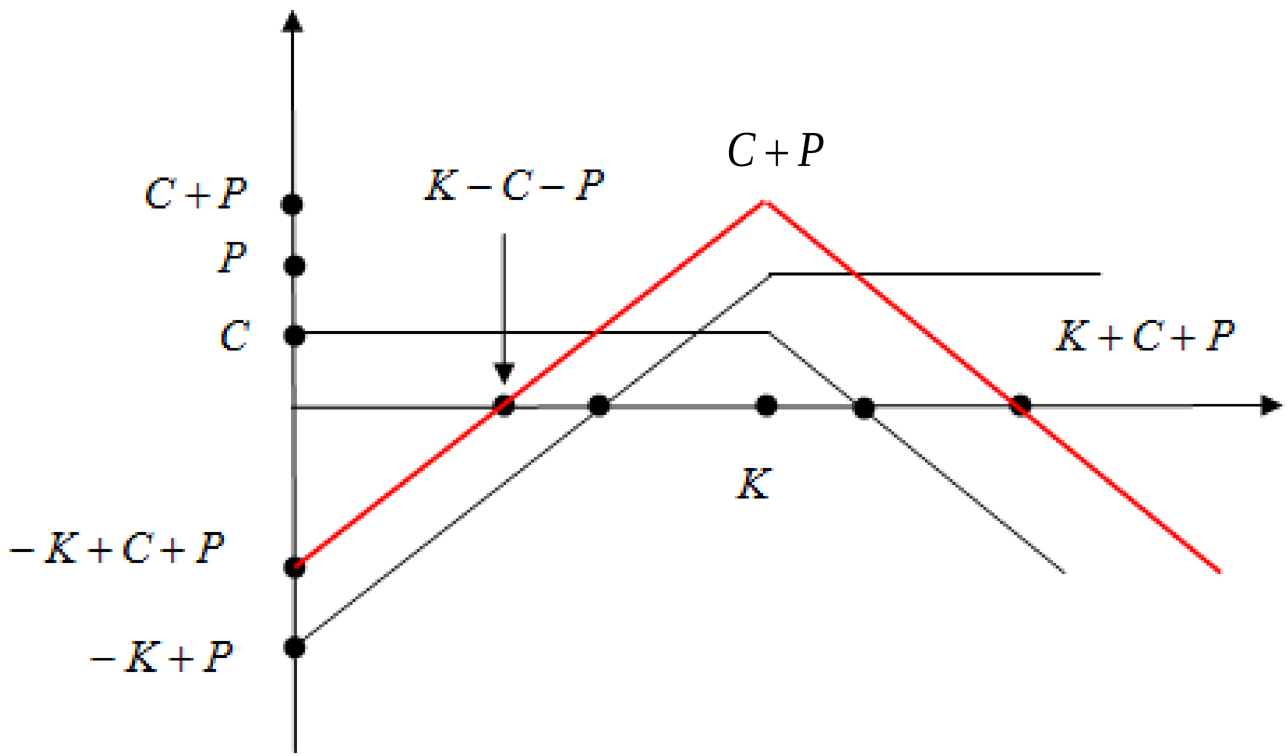
$-C-P$

$K = 40 \quad C = 3$

$K = 40 \quad P = 5$

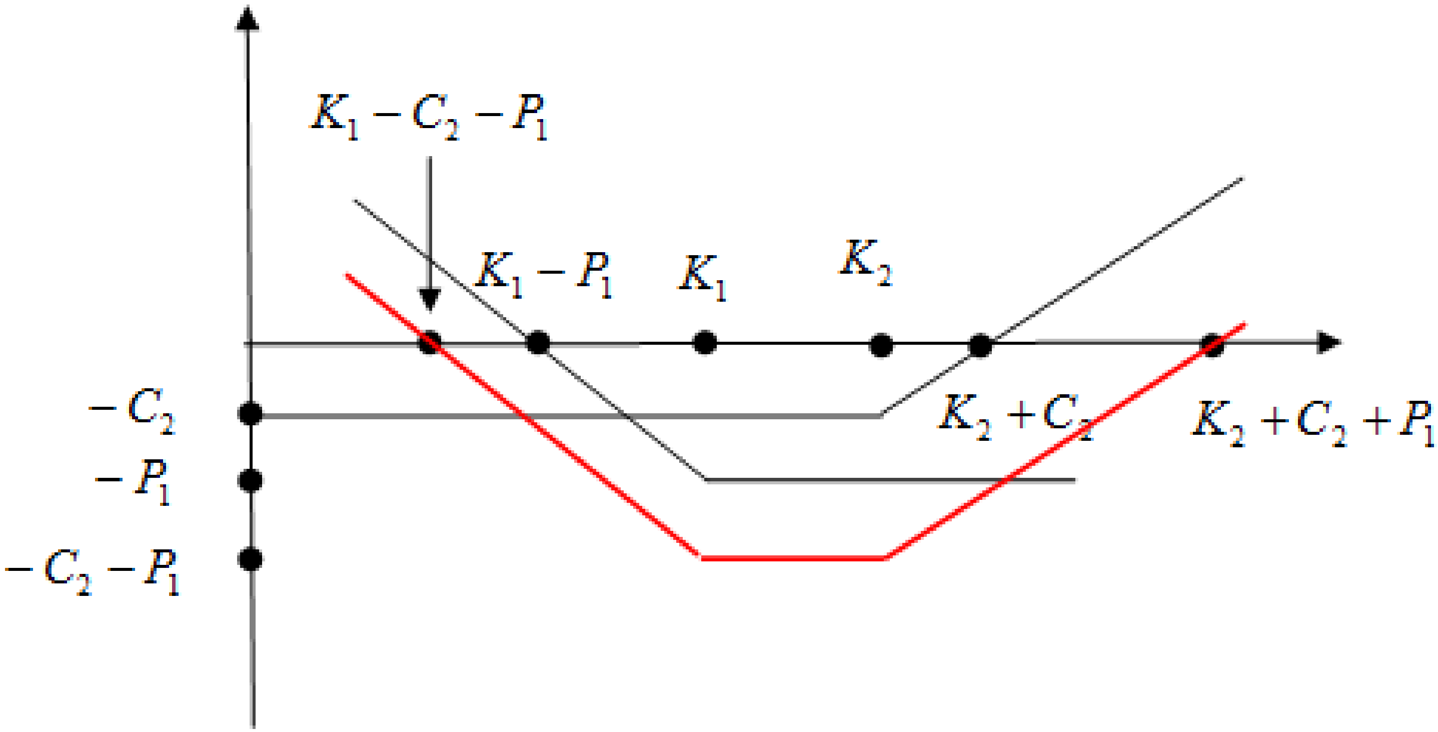
$\max\{S_T - K, 0\} - C \qquad \max\{K - S_T, 0\} - P$

Short straddle - short call + short put

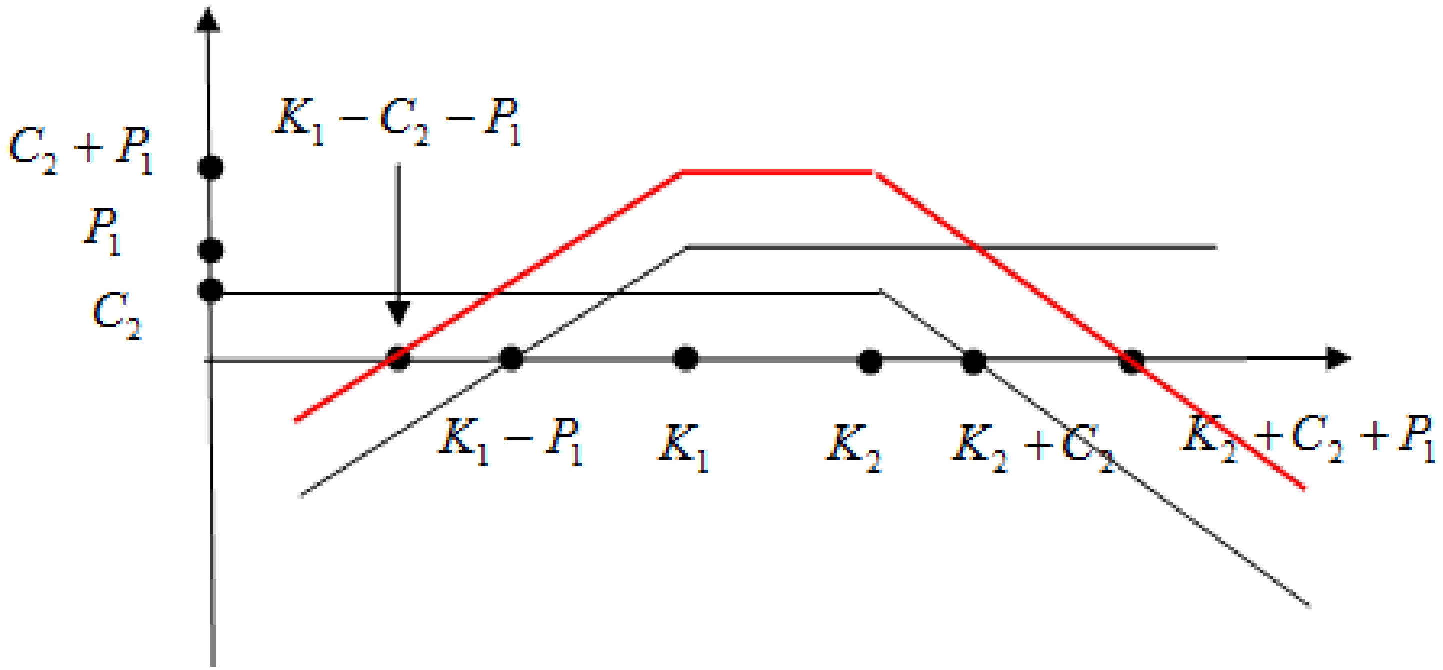


$$\min\{K-S_T,0\}+C \qquad \min\{S_T-K,0\}+P$$

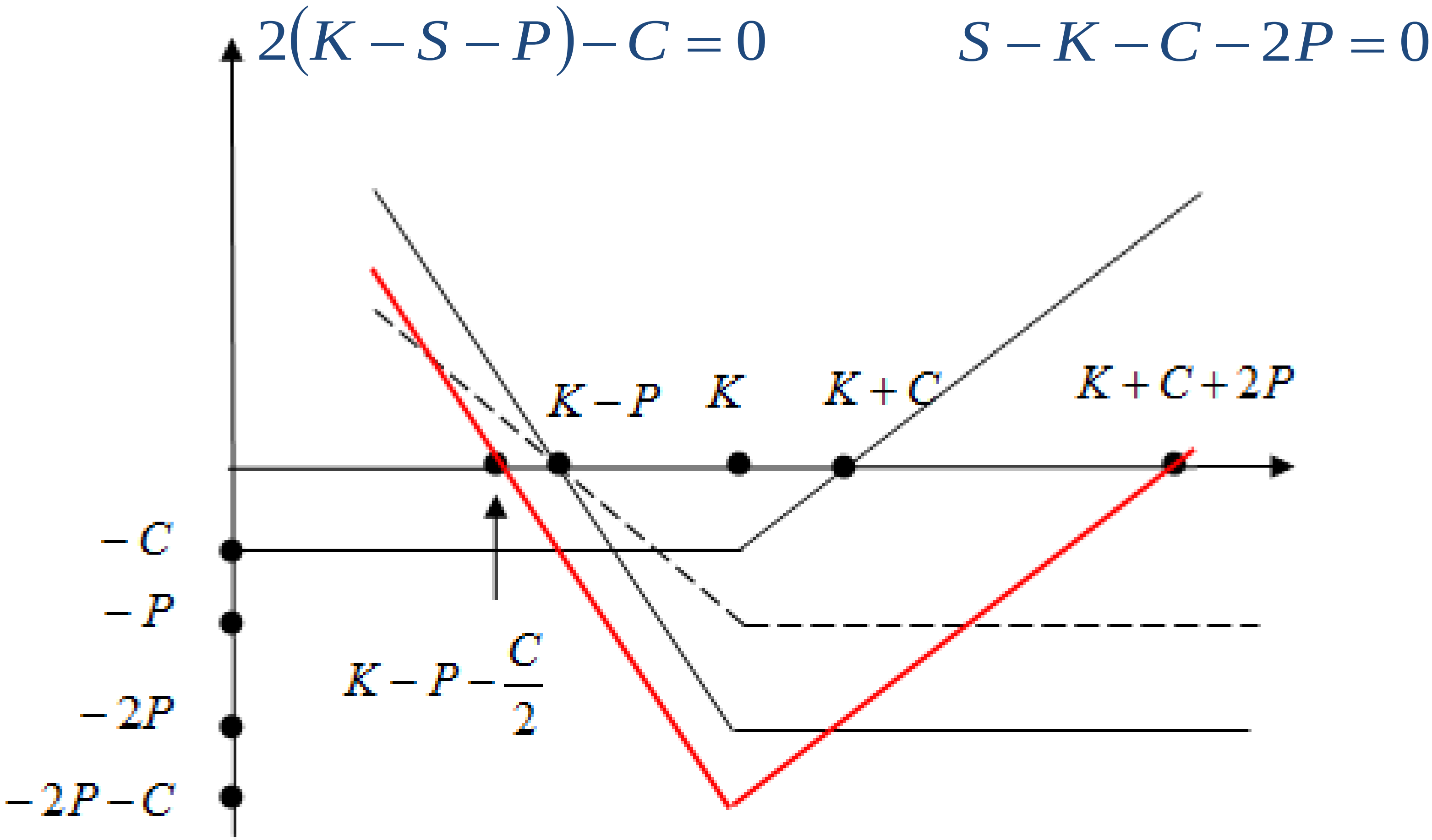
Long strangle - long call + long put



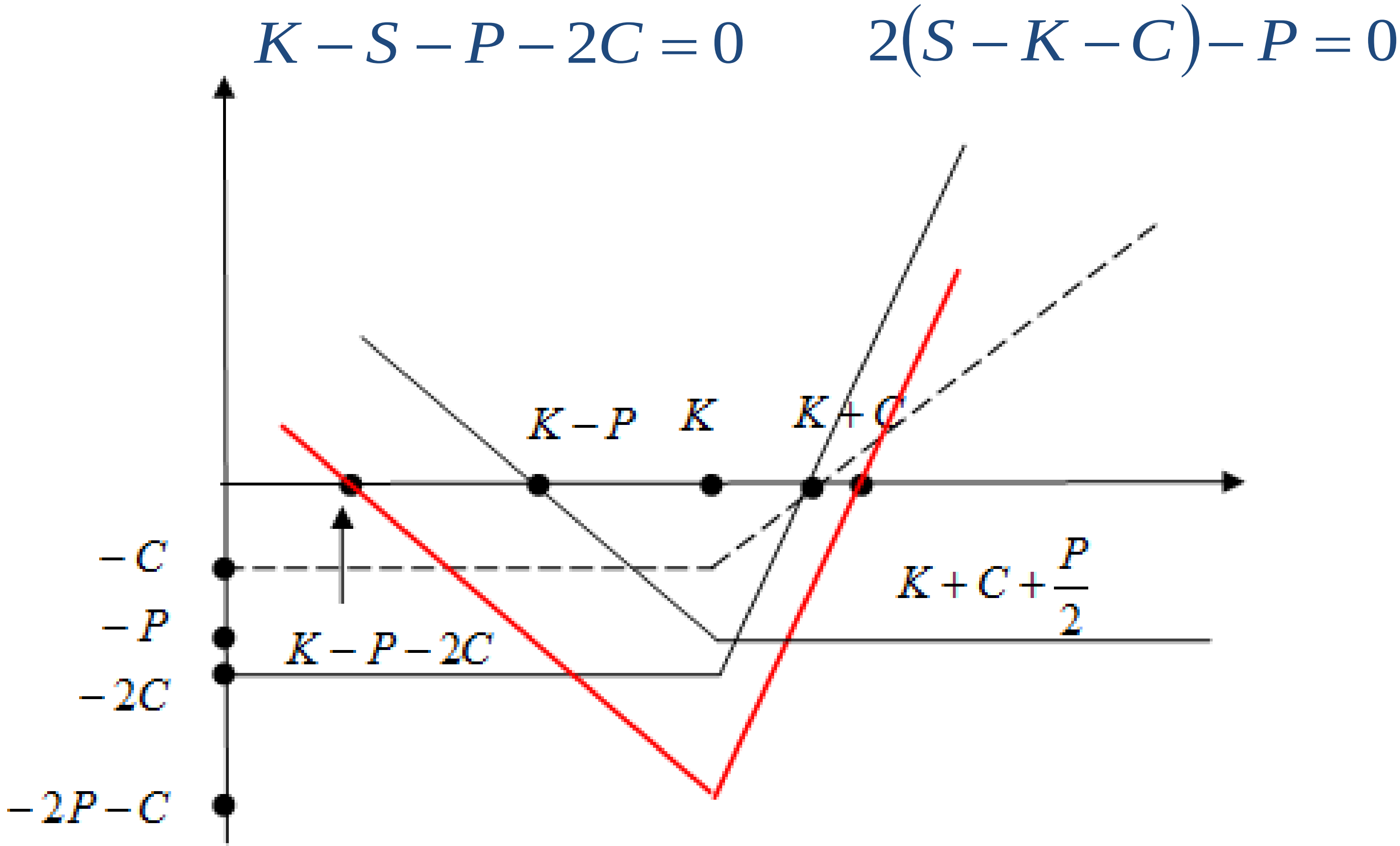
Short strangle - short call + short put



Strip



Strap



The Black-Scholes model - assumptions

- Markets are efficient (market movements cannot be predicted)
- There are no transaction costs in buying the option
- The risk-free rate and volatility of the underlying asset are known and constant
- The returns on the underlying are normally distributed
- The option is European and can only be exercised at expiration
- No dividends are paid out during the life of the option

Pricing options in the Black-Scholes framework

$$C(S,t) = S(t)\cdot N(d_1) - K\cdot e^{-r(T-t)}N(d_2)$$

$$P(S,t) = -S(t)\cdot N(-d_1) + K\cdot e^{-r(T-t)}N(-d_2)$$

$$d_1 = \frac{ln(S(t)/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

- T – an expiration time, r – a risk-free interest rate, σ – a stock's volatility (a standard deviation), K – a strike price, $S(t)$ – price of the stock at t, $N(x)$ – the cumulative standard normal distribution (see a standard normal distribution table).

Example

- You are given:
- The Back-Scholes framework holds
- The stock is currently selling for 50 PLN
- The option will expire in 5 months with a strike price of 48 PLN
- The stock’s volatility is 20%
- The continuously compounded risk-free interest rate is 10%
- Calculate the price of the European call and put options on a nondividend-paying stock.

Example

$$S = 50 \quad K = 48 \quad \sigma = 0.2 \quad r = 0.1 \quad T = 5/12$$

$$d_1 = \frac{ln(S(t)/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

$$d_1 = \frac{ln(50/48) + \left(0.1 + \frac{(0.2)^2}{2}\right)\frac{5}{12}}{0.2\sqrt{5/12}} = 0.70350414$$

$$d_2 = d_1 - \sigma\sqrt{T - t} = 0.5744047$$

Example

$$N\left(d_1\right) = 0.759123283 \qquad N\left(d_2\right) = 0.717146216$$

$$C\left(S,t\right) = S\left(t\right) \cdot N\left(d_1\right) - K \cdot e^{-r\left(T-t\right)} N\left(d_2\right)$$

$$C\left(S,t\right) = 50 \cdot 0.7591 - 48 \cdot e^{-0.1 \cdot \frac{5}{12}} \cdot 0.7171 = \boxed{4.937968}$$

$$N\left(-d_1\right) = 0.240665495 \qquad N\left(-d_2\right) = 0.2827498$$

$$P\left(S,t\right) = -S\left(t\right) \cdot N\left(-d_1\right) + K \cdot e^{-r\left(T-t\right)} N\left(-d_2\right) = \boxed{0.984835}$$

Example

	European call option	**European put option**
The Black-Scholes model	4.937968	0.984835
Binominal option tree (continuously compounded interest)	4.932961	0.988305

The Merton model

$$C(S,t) = S(t)\cdot e^{-\delta(T-t)}\cdot N(d_1) - K\cdot e^{-r(T-t)}N(d_2)$$

$$P(S,t) = -S(t)\cdot e^{-\delta(T-t)}\cdot N(-d_1) + K\cdot e^{-r(T-t)}N(-d_2)$$

$$d_1 = \frac{ln(S(t)/K) + (r - \delta + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

- T – an expiration time, r – a risk-free interest rate, σ – a stock's volatility (a standard deviation), K – a strike price, $S(t)$ – price of the stock at t, $N(x)$ – the cumulative standard normal distribution δ – the dividend yield of the stock.

Example (the Merton model)

- The stock is currently selling for 90 PLN,
- The stock's volatility is 20%
- The strike price is 100 PLN, risk-free interest rate is10%
- The option expired at 3 months.
- The stock pays dividend continuously at a rate proportional to its price. The dividend yield is 3%.

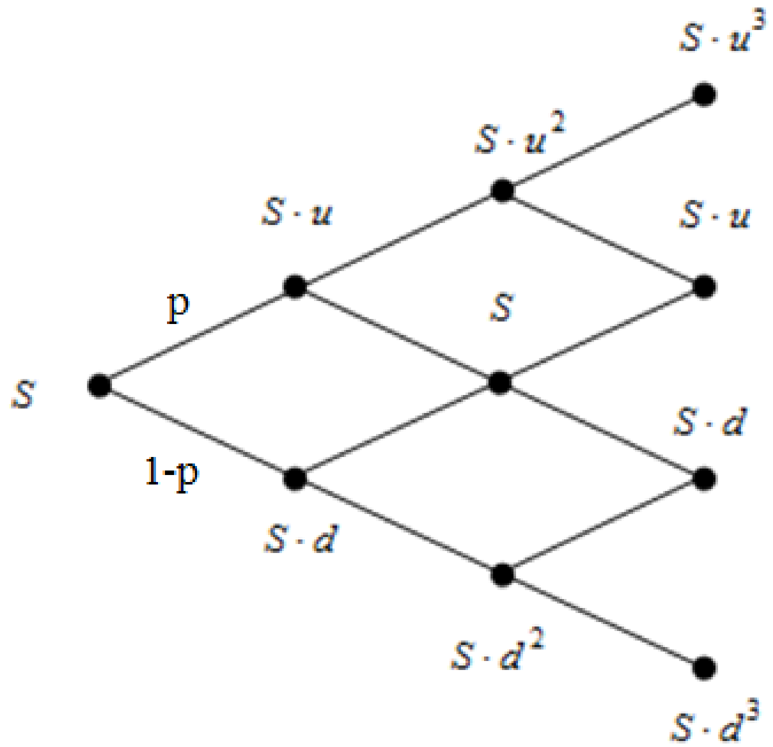
$$S = 90 \quad K = 100 \quad \sigma = 0.2 \quad r = 0.1 \quad T = 3/12 \quad \delta = 0.03$$

$$C(S,t) = 1.00345 \qquad\qquad P(S,t) = 9.17748$$

Appendix (u, d and p)

$$S_n^k = S \cdot u^k \cdot d^{n-k}$$

$$\binom{n}{k} \cdot p^k (1-p)^{n-k}$$



Binomial coefficient

$$\binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}$$

$n!$ - factorial of n

$$S_n^k = S \cdot u^k \cdot d^{n-k}$$

$$ln\, S_n^k = ln(S) + k \cdot ln(u) + (n-k) \cdot ln(d)$$

$$S_0 = 1$$

$$ln\, S_n^k = k \cdot ln(u) + (n-k) \cdot ln(d)$$

$S_0 = 1$ S_1

$$E(\ln S_1) = p \cdot \ln u + (1-p) \cdot \ln d$$

$$var(\ln S_1) = \sum_i p_i [\ln S_i - E(\ln S_1)]^2$$

$$var(\ln S_1) = p[\ln S_u - p \cdot \ln u - (1-p)\ln d]^2 + (1-p)[\ln S_d - p \cdot \ln u - (1-p)\ln d]^2$$

$$var(\ln S_1) = p[\ln u - p \cdot \ln u - (1-p)\ln d]^2 + (1-p)[\ln d - p \cdot \ln u - (1-p)\ln d]^2$$

$$var(ln\,S_1) = p[ln\,u - p \cdot ln\,u - (1-p)ln\,d]^2$$
$$+ (1-p)[ln\,d - p \cdot ln\,u - (1-p)ln\,d]^2$$

$$var(ln\,S_1) = p(1-p)^2[ln\,u - ln\,d]^2 + (1-p)p^2[ln\,u - ln\,d]^2$$

$$var(ln\,S_1) = [p(1-2p+p^2) + (1-p)p^2] \cdot [ln\,u - ln\,d]^2$$

$$var(ln\,S_1) = p \cdot (1-p) \cdot [ln\,u - ln\,d]^2$$

$$\nu = E\left[ln\left(\frac{S_T}{S_0}\right)\right]$$

$$S_T = S_0 e^{\nu \cdot T}$$

$$\nu \cdot T = ln\left(\frac{S_T}{S_0}\right)$$

$$\sigma^2 = var\left[ln\left(\frac{S_T}{S_0}\right)\right]$$

$$\nu \cdot \Delta t = E\left(ln\left(S_1\right)\right)$$

$$\sigma^2 = \mathrm{var}\left[\ln\left(S_1\right)\right]$$

$$E(\ln S_1) = p \cdot \ln u + (1-p) \cdot \ln d = \nu \cdot \Delta t$$

$$var(\ln S_1) = p \cdot (1-p) \cdot [\ln u - \ln d]^2 = \sigma^2 \cdot \Delta t$$

$$\begin{cases} p \cdot U + (1-p) \cdot D = \nu \cdot \Delta t \\ p \cdot (1-p) \cdot (U-D)^2 = \sigma^2 \cdot \Delta t \\ D = -U \end{cases}$$

$$U = \ln u$$

$$D = \ln d$$

$$U = -D$$

$$\ln u = -\ln d$$

$$u = 1/d$$

$$\begin{cases} p \cdot U + (1-p) \cdot D = \nu \cdot \Delta t \\ p \cdot (1-p) \cdot (U-D)^2 = \sigma^2 \cdot \Delta t \\ D = -U \end{cases}$$

$$\begin{cases} p \cdot U - (1-p) \cdot U = \nu \cdot \Delta t \\ 4p \cdot (1-p) \cdot U^2 = \sigma^2 \cdot \Delta t \end{cases}$$

$$\begin{cases} (2p-1) \cdot U = \nu \cdot \Delta t \\ U^2 = \sigma^2 \cdot \Delta t + (\nu \cdot \Delta t)^2 \end{cases}$$

$$\begin{cases} U^2 = \dfrac{(\nu \cdot \Delta t)^2}{(2p-1)^2} \\ U^2 = \sigma^2 \cdot \Delta t + (\nu \cdot \Delta t)^2 \end{cases}$$

$$\frac{(\nu \cdot \Delta t)^2}{(2p-1)^2} = \sigma^2 \cdot \Delta t + (\nu \cdot \Delta t)^2$$

$$\frac{1}{(2p-1)^2} = \frac{\sigma^2 \cdot \Delta t}{(\nu \cdot \Delta t)^2} + 1$$

$$\frac{1}{(2p-1)} = \sqrt{\frac{\sigma^2 \cdot \Delta t}{(\nu \cdot \Delta t)^2} + 1}$$

$$p = \frac{1}{2}\left(1 + 1 \Big/ \sqrt{\frac{\sigma^2 \cdot \Delta t}{(\nu \cdot \Delta t)^2} + 1}\right)$$

$$\Delta t << 1 \qquad \frac{\sigma^2}{\nu^2 \cdot \Delta t} >> 1$$

$$p = \frac{1}{2}\left(1 + \frac{\nu \cdot \sqrt{\Delta t}}{\sigma}\right)$$

$$(2p-1)\cdot U = \nu \cdot \Delta t$$

$$ln\, u = \frac{\nu \cdot \Delta t}{2p-1}$$

$$\frac{1}{(2p-1)} = \sqrt{\frac{\sigma^2 \cdot \Delta t}{(\nu \cdot \Delta t)^2} + 1}$$

$$ln\, u = \nu \cdot \Delta t \cdot \sqrt{\frac{\sigma^2}{\nu^2 \cdot \Delta t} + 1}$$

$$\Delta t << 1 \qquad \frac{\sigma^2}{\nu^2 \cdot \Delta t} >> 1$$

$$ln\, u = \sigma \cdot \sqrt{\Delta t}$$

$$u = e^{\sigma \cdot \sqrt{\Delta t}}$$

$$d = e^{-\sigma \cdot \sqrt{\Delta t}}$$