Financial Mathematics

Lecture 1-2

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http://prawo.uni.wroc.pl/user/12141/students-resources

Syllabus

Time value of money

- Percentage change. Interest rate. Simple interest, compound interest, compounding frequency, compounding agreement, continuously compounded interest, equation of value, future value, present value, discount factor, effective rate, average interest rate.

Syllabus

- **Annuities**: Present and future value of an annuity, annuity payment, annuity-immediate, annuity due, level payment annuity, non-level annuities, perpetuity.

- **Loans**: Principal, interest, payment amount, payment period. Long-term loan repayment methods. Equal principal payments. Equal total payments. Other loan repayment methods. Loan amortization schedule.

Syllabus

- **Money market instruments**: treasury bills and certificates of deposit. Pricing and quotation, rate of return, discount yield.

- **Capital market instruments**: treasury bonds and stocks.

Syllabus

- **Analysis of financial derivatives**:
- Valuation of forward and futures contracts. The fair price of forward contract:
  - the underlying asset pays no income,
  - the underlying asset pays predictable income,
  - the underlying asset pays continuous dividend yields
- Swaps. Interest rate swap and currency swap

Syllabus

- **Options**:
- The binomial option pricing model
- Binomial model for a non-dividend paying stock and for a known dividend paying stock
- Black-Scholes model
- Option strategies

Recommended Reading

- Kevin J. Hastings, *Introduction to Financial Mathematics*, CRC Press, 2015.

Percentage

- Convert percentage to decimal – divide percentage amount by 100

$$20\% \rightarrow \frac{20}{100} = 0.2$$

- Decimal to percentage – multiply decimal by 100

0.02 to %

$$0.02 \cdot 100\% = 2\%$$

- Percentage of a quantity

How much is **13% of 100 PLN?**

$$0.13 \cdot 100 PLN = 13 PLN$$

Calculating the discount and the new price

- The price of an item is discounted, or marked down, by *r*% (*r*% decrease from X PLN)

$$discount = \frac{r}{100} \cdot original\ price$$

$$new\ price = original\ price - discount$$

$$discount = original\ price - new\ price$$

Calculating the discount and the new price

- How much is saved if a 15% discount is offered on an item marked 20 PLN? What is the new discounted price of this item?

$$0.15 \cdot 20PLN = 3PLN$$

$$20PLN - 0.15 \cdot 20PLN = 17PLN$$

Calculating the increase and the new price

- The price of an item is increased, or marked up, by *r*% (*r*% increase from X PLN)

$$increase = \frac{r}{100} \cdot original\ price$$

$$new\ price = original\ price + increase$$

$$increase = new\ price - original\ price$$

Calculating the increase and the new price

- How much is added if a 15% increase is applied to an item marked 20 PLN? What is the new increased price of this item?

$$0.15 \cdot 20PLN = 3PLN$$

$$20PLN + 0.15 \cdot 20PLN = 23PLN$$

One quantity expressed as a percentage of another quantity

- Express 15 PLN as a percentage of 200 PLN (What percentage of 200 is 15?)

$$\frac{15}{200} \cdot 100\% = 0.075 \cdot 100\% = 7.5\%$$

- 7.5% of 200 PLN = 15 PLN

Calculating the percentage change

- Given the original price and the new price of an item, we can work out the **percentage change.** To do this, the amount of the decrease or increase is determined and then converted to a percentage of the original price.

$$percentage\,discount = \frac{discount}{original\ \ price} \cdot 100\%$$

$$percentage\,increase = \frac{increase}{original\ \ price} \cdot 100\%$$

Calculating the percentage change

- If the price of an item is reduced from 200 PLN to 160 PLN, what percentage discount has been applied?

$$\frac{200-160}{200}\cdot 100\% = 0.2\cdot 100\% = 20\%$$

- If the price of an item is increased from 200 PLN to 260 PLN, what percentage increase has been applied?

$$\frac{260-200}{200}\cdot 100\% = 0.3\cdot 100\% = 30\%$$

Calculating the original price

- When a r% discount has been applied

$$original\ \ price = \frac{100}{100 - r} \cdot new\ price$$

- When a r% increase has been applied

$$original\ \ price = \frac{100}{100 + r} \cdot new\ price$$

Calculating the original price

- Find the original price of the item that has been:

a. marked down by 10%, now priced 90 PLN

$$original\ \ price = \frac{100}{100-10} \cdot 90 = 100$$

b. marked up by 10%, now priced 90 PLN

$$original\ \ price = \frac{100}{100+10} \cdot 90 = 81.82$$

Calculating the new price

- When a *r*% discount has been applied

$$new\ price = \frac{100 - r}{100} \cdot original\ price$$

- When a *r*% increase has been applied

$$new\ price = \frac{100 + r}{100} \cdot original\ price$$

Exercise 1

1. Calculate the amount of the discount
a) 14% discount on 98 PLN
b) 1.5% discount on 400 PLN

2. Calculate the amount of the increase
a) 0.3% increase on 10 000 PLN
b) 5% increase on 2 PLN

3. Calculate the following as percentages
a) 18.45 PLN of 150 PLN
b) 0.2 PLN of 4 PLN

Exercise 1

4. Calculate the new discounted prices for each of the following

a) 164 PLN discounted by 4.5%

b) 20 000 PLN discounted by 43%

5. Calculate the new increased prices for each of the following

a) 2.5 PLN marked up by 30%,

b) 1000 PLN marked up by 2.5%

Exercise 1

6. Find the original price of the item that has been:

a. marked down by 5%, now priced 80 PLN,

b. marked down by 15%, now priced 50 PLN,

c. marked up by 5%, now priced 80 PLN,

d. marked up by 15%, now priced 50 PLN.

- Principle – original amount invested or borrowed
- Interest – the amount of interest earned
- Rate – percentage rate of interest to be earned per annum
- Term – duration of loan/investment in years
- Amount of money
- Number of compounding periods

- Compounding – determining the future value by the use of compounding interest, that is, interest on interest, period by period.

- The frequency of compounding: annual, semi-annual, monthly, daily, continuous

- Continuous compounding is a term used when taking the limit of the interest rate as the period of time compounding approaches to 0.

Simple interest

$$K_1 = K_0 + rK_0 = (1 + r)K_0$$

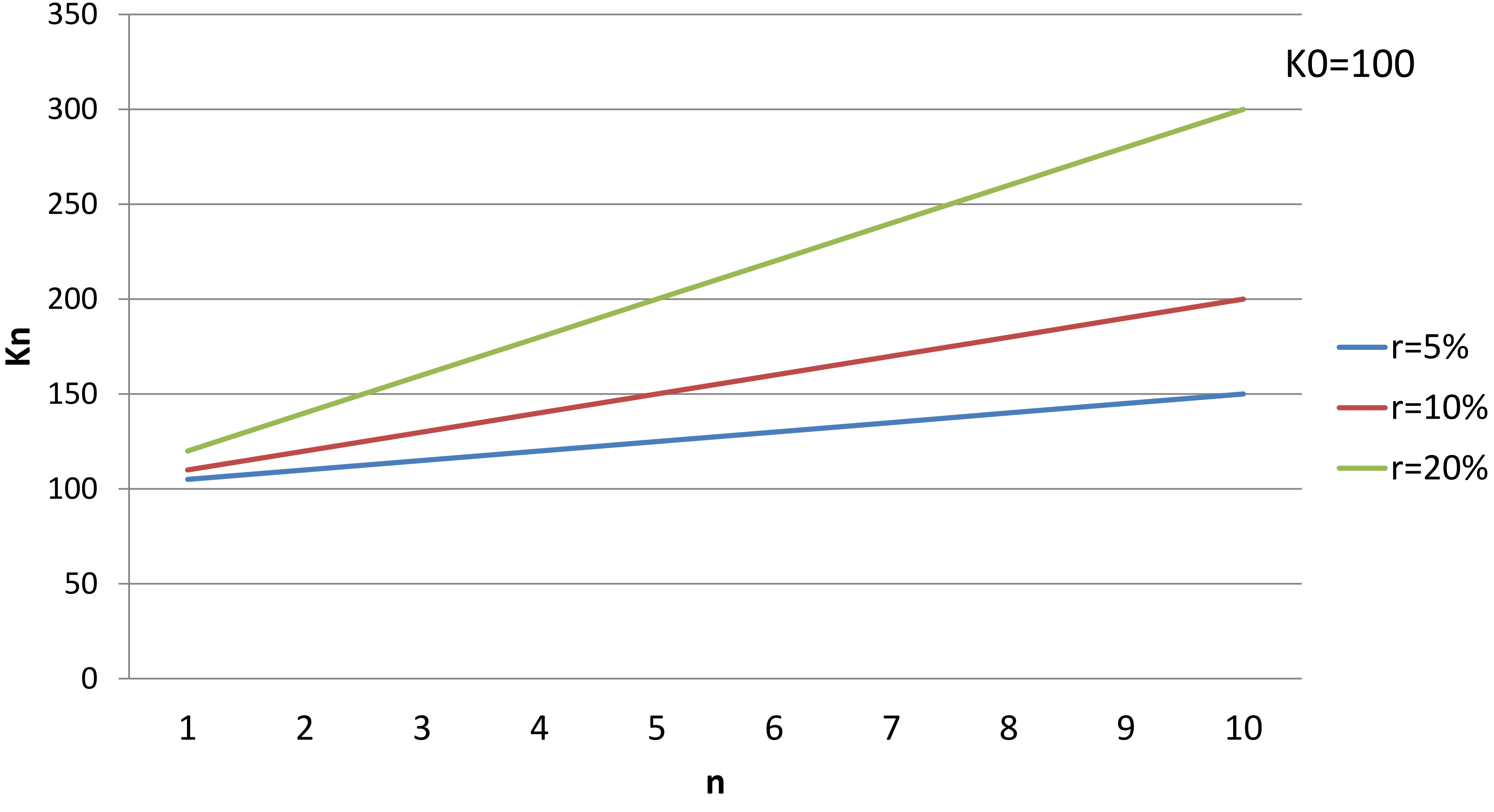
$$K_2 = K_1 + rK_0 = (1 + 2r)K_0$$

$$K_n = (1 + n \cdot r)K_0$$

$$r = \frac{K_n - K_0}{n \cdot K_0}$$

$$n = \frac{K_n - K_0}{r \cdot K_0}$$

Simple interest



Compound interest

$$K_1 = K_0 + rK_0 = K_0 \cdot (1 + r)$$

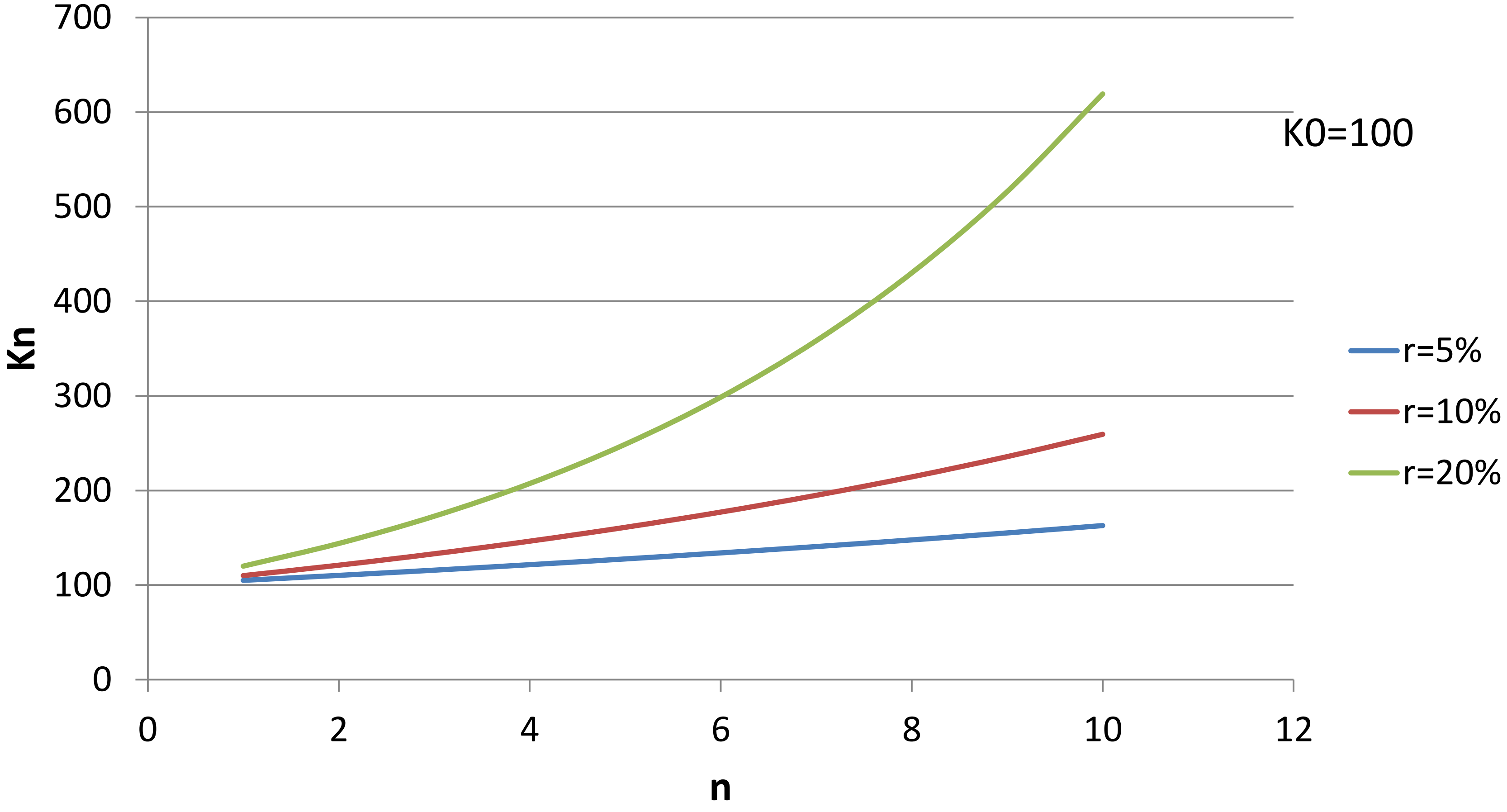
$$K_2 = K_1 + rK_1 = K_0 \cdot (1 + r)^2$$

$$K_n = K_0 \cdot (1 + r)^n$$

$$r = \sqrt[n]{\frac{K_n}{K_0}} - 1$$

$$n = \frac{\ln(K_n / K_0)}{\ln(1 + r)}$$

Compound interest



Compound interest (the beginning of the period)

$$K_1 = K_0 + K_0 \cdot r + K_0 \cdot r^2 + \ldots$$

$$K_1 = K_0\left(1 + r + r^2 + \ldots\right)$$

$$K_1 = K_0 \cdot (1 - r)^{-1} \qquad |r| < 1$$

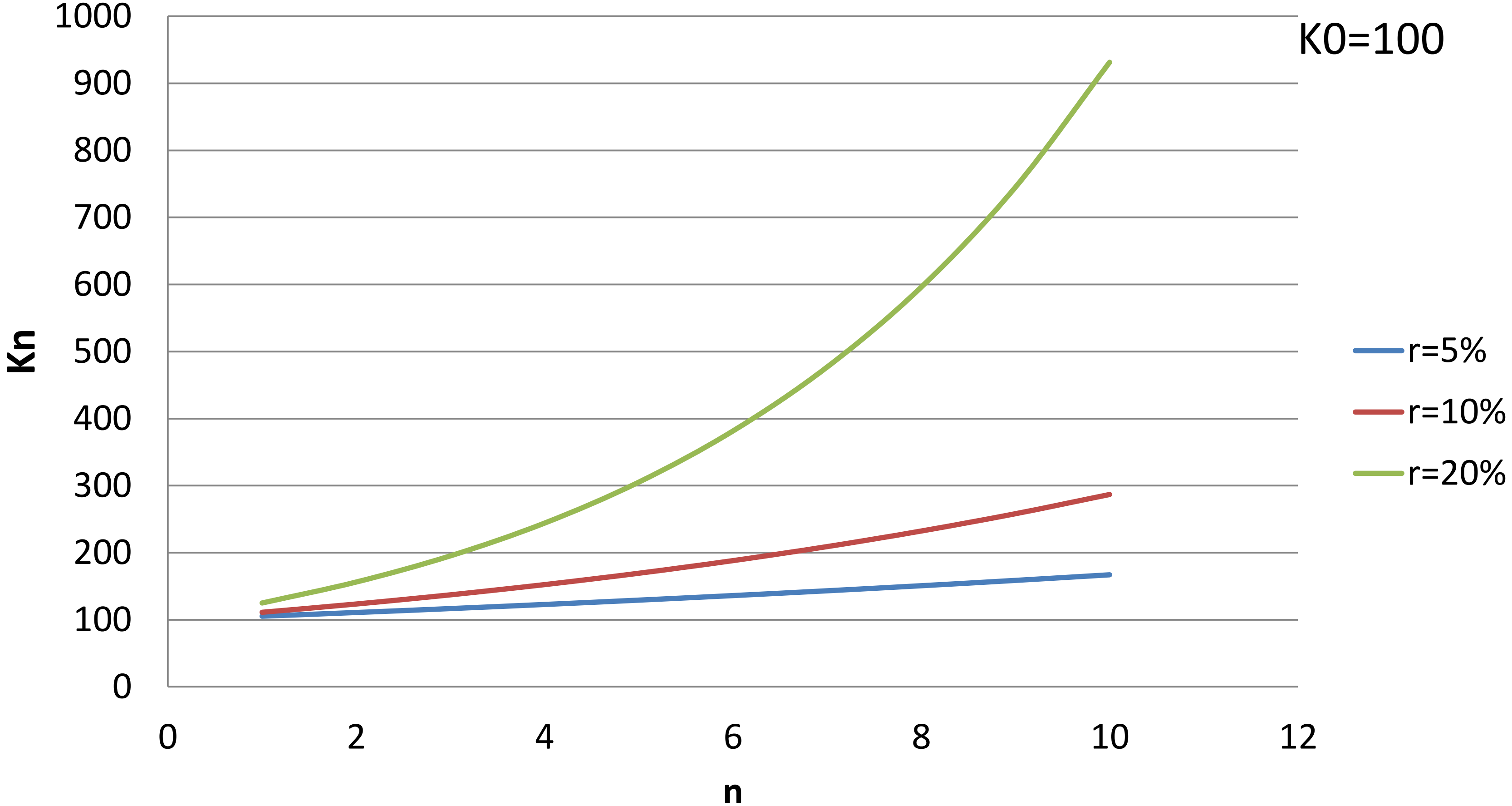
$$K_2 = K_1 \cdot (1 - r)^{-1} = K_0 \cdot (1 - r)^{-2}$$

$$K_n = K_0 \cdot (1 - r)^{-n}$$

$$r = 1 - \sqrt[n]{\frac{K_0}{K_n}}$$

$$n = \frac{\ln\left(K_0 / K_n\right)}{\ln\left(1 - r\right)}$$

Compound interest
(the beginning of the period)



Continuously compounded interest

$$K_n = K_0 \cdot e^{n \cdot r}$$

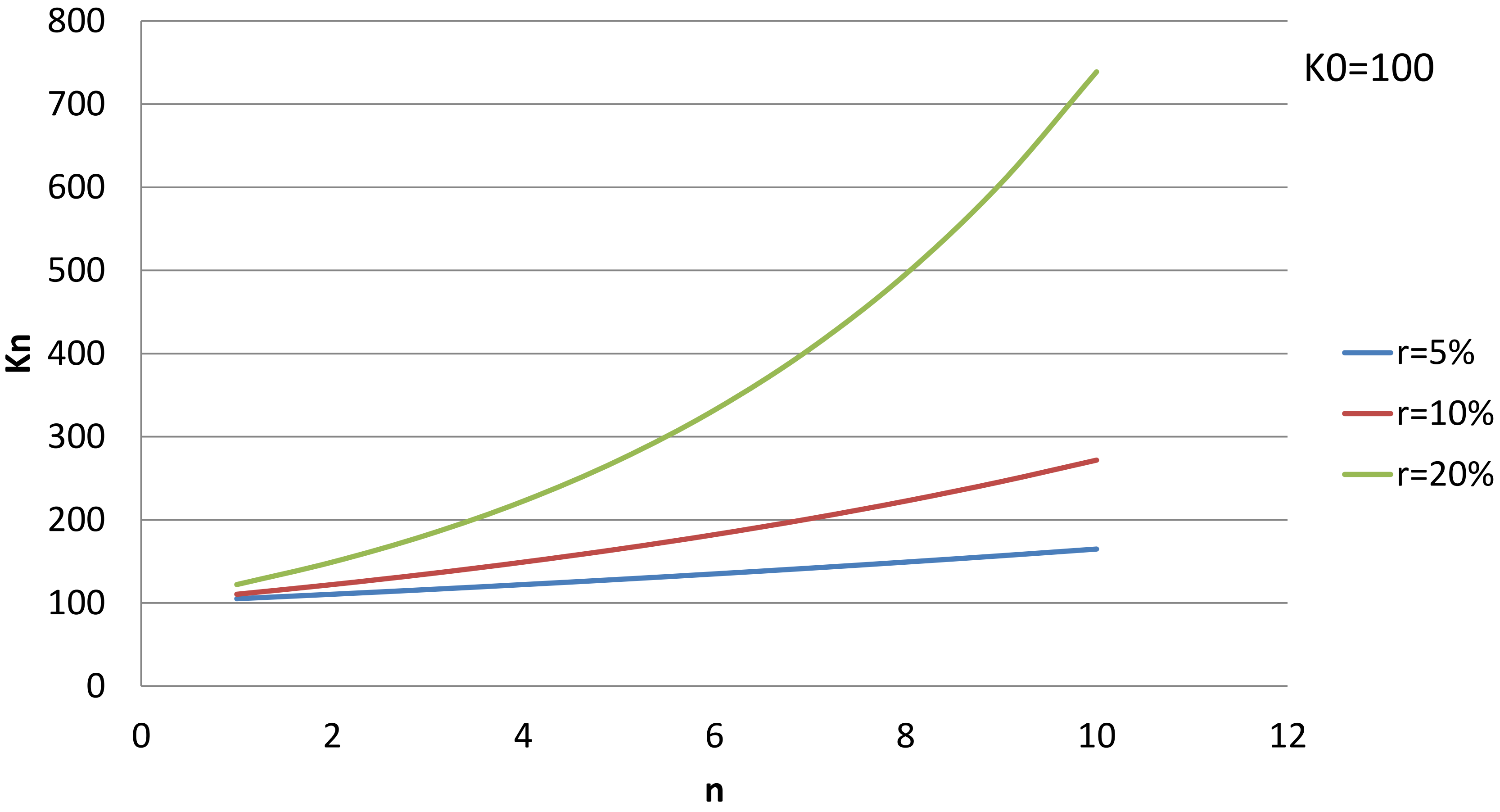
$$\lim_{m \to \infty} K_0 \cdot (1 + \frac{r}{m})^{n \cdot m} = K_0 \cdot e^{n \cdot r}$$

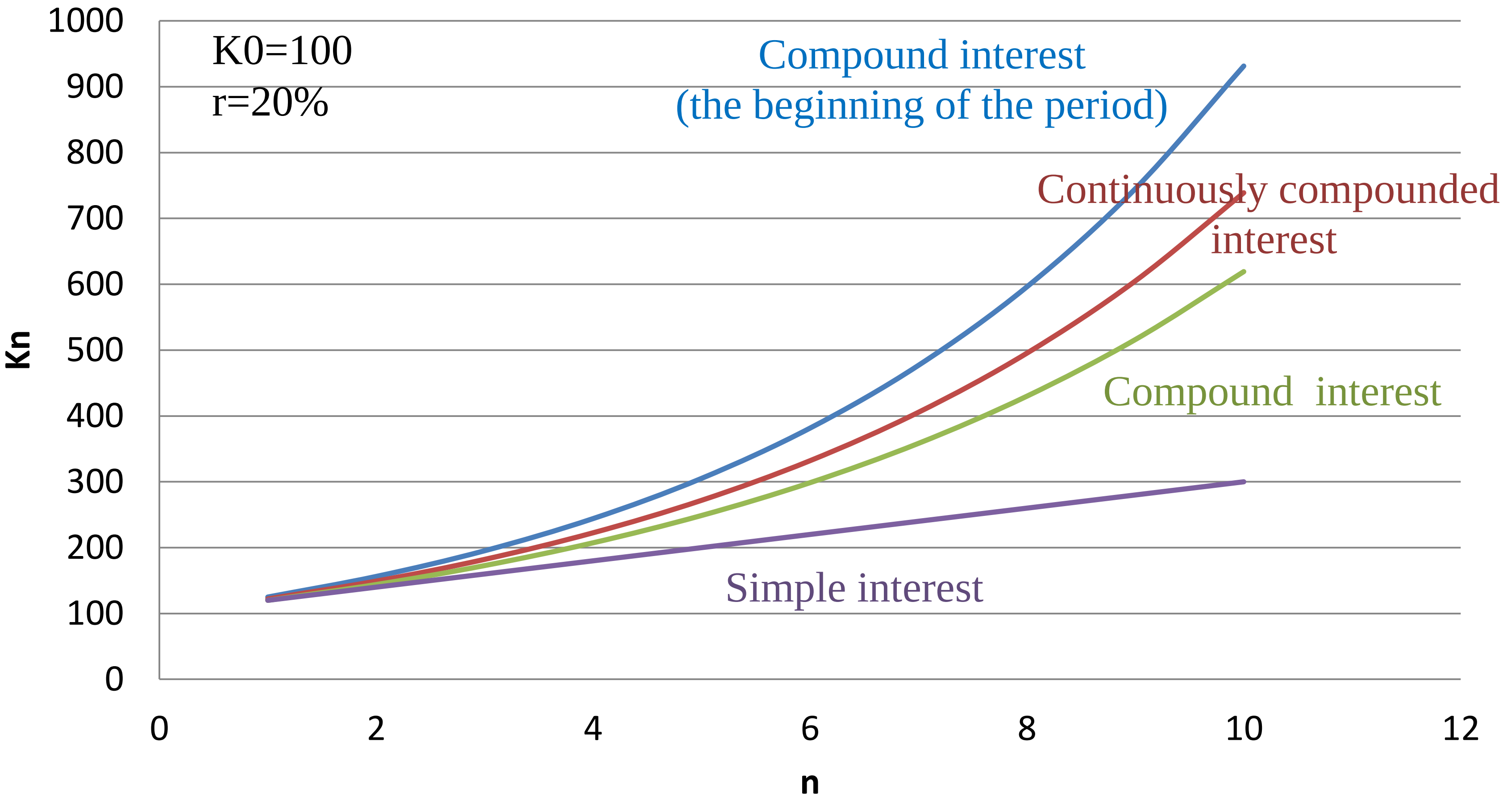
$$\lim_{m \to \infty} K_0 \cdot (1 - \frac{r}{m})^{-n \cdot m} = K_0 \cdot e^{n \cdot r}$$

$$r = \frac{\ln(K_n / K_0)}{n}$$

$$n = \frac{\ln(K_n / K_0)}{r}$$

Continuously compounded interest





K0=100
r=20%
Compound interest
(the beginning of the period)
Continuously compounded
interest
Compound interest
Simple interest
1000
900
800
700
600
500
400
300
200
100
0
Kn
0
2
4
6
8
10
12
n

Simple interest – examples

- How much interest is earned if 1000 PLN is invested at 4% per annum simple interest for 5 years?

$$I = 1000 \cdot 0.04 \cdot 5 = 200$$

- Find the total amount owed on a loan of 20000 PLN at 5% per annum simple interest at the end of 2 years.

$$K_2 = 20000(1 + 2 \cdot 0.05) = 22000$$

Simple interest – examples

- A sum of 5 000 PLN was invested at a simple interest rate for 2 years. The total value of investment at the end of the 2 years is 10 000 PLN. Find the quarterly interest rate.

$$K_8 = K_0(1+8\cdot r) \qquad \frac{10}{5} - 1 = 8r \qquad r = 12.5\%$$

- Find the simple interest rate if a principal increases seven times in 10 years.

$$7K_0 = K_0(1+10\cdot r) \qquad r = 60\%$$

Compound interest – examples

- Determine the amount of money accumulated after 3 years if 6 000 PLN is invested at an interest rate of 5% per annum, compounded monthly. Determine the amount of interest earned.

$$K_3 = 6000\left(1+\frac{0.05}{12}\right)^{36} = 6968.83$$

$$I = 6968.83 - 6000 = 968.83$$

Compound interest – examples

- How much money should be invested at 8% per annum compound interest, compounding quarterly if 10 000 PLN is needed in 4 years time?

$$K_0 = \frac{10000}{\left(1+\frac{0.08}{4}\right)^{16}} = 7284.46$$

Compound interest – examples

- How many months does it take for 4 000 PLN to accumulate to 10 000 PLN under 9% p.a. compound interest (continuously compounded interest)?

$$10000 = 4000\left(1+\frac{0.09}{12}\right)^{n}$$

$$10000 = 4000e^{n\frac{0.09}{12}}$$

$$\ln(2.5) = n \cdot \ln(1.0075)$$

$$\ln(2.5) = n \cdot 0.0075$$

$$n = 122.63$$

$$n = 122.17$$

Continuously compounded interest – example

- Determine the amount of money accumulated after 3 years if 6 000 PLN is invested at an interest rate of 5% per annum, continuously compounded interest. Determine the amount of interest earned.

$$K_3 = 6000e^{3 \cdot 0.05} = 6971.01$$

$$I = 971.01$$

Simple and compound interest – examples

- Suppose that a capital of 400 PLN earns 150 PLN of interest in 6 years. What was the interest rate if compound interest is used? What if simple interest is used?

$$550 = 400(1+r)^6 \qquad r = 5.451\%$$

$$550 = 400(1+6r) \qquad r = 6.25\%$$

Simple and compound interest – examples

- A bank offers various investment possibilities to a customer wishing to invest 25 000 PLN for 10 years. Calculate the final amount for each of the following

a. Simple interest rate at 15% per annum,
b. Compound interest at 11% per annum, calculated annually,
c. Compound interest at 10.5% per annum, calculated semi-annually,
d. Compound interest at 10% per annum, calculated quarterly,
e. Compound interest at 9.5% per annum, calculated monthly,
f. Compound interest at 9% per annum, calculated daily.

Which investment would you recommend?

Simple and compound interest – examples

- A bank offers various investment possibilities to a customer wishing to invest 25 000 PLN for 10 years. Calculate the final amount for each of the following

a. Simple interest rate at 15% per annum, (62 500 PLN)
b. Compound interest at 11% per annum, calculated annually, (70 985.52 PLN)
c. Compound interest at 10.5% per annum, calculated semi-annually, (69 563.61 PLN)
d. Compound interest at 10% per annum, calculated quarterly, (67 126.6 PLN)
e. Compound interest at 9.5% per annum, calculated monthly, (64 401.38 PLN)
f. Compound interest at 9% per annum, calculated daily (365). (61 483.26 PLN)

Which investment would you recommend?

Discounting

- The discount factor is the amount of money one needs to invest to get one unit of capital after one time unit.

- Simple discounting
- Compound discounting

Discounting - example

- What is a present value of 10 000 PLN due in a month (a quarter) assuming 9% p.a. simple discount?

$$10000 \cdot \left(1 - \frac{0.09}{12}\right) = 992.5$$

$$10000 \cdot \left(1 - \frac{0.09}{4}\right) = 977.5$$

The frequency of compounding

- The stated interest rate can deviate significantly from the true interest rate.
- Effective interest rate
- Example a 20% annual interest rate

$$r_{ef} = \left(1 + \frac{r}{m}\right)^m - 1$$

Frequency	Effective annual rate	m
Annual	20%	1
Semi-Annual	21.0000%	2
Quarterly	21.5506%	4
Monthly	21.9391%	12
Daily	22.1335% 22.1336%	360/365
Continuous	22.1403%	

$$r_{ef} = e^r - 1$$

Compounding agreement

	Compounding annually	Compounding semi-annually	Compounding quarterly	Compounding monthly
Annual rate	r	$r_{ef} = \left(1+\frac{r}{2}\right)^2 - 1$	$r_{ef} = \left(1+\frac{r}{4}\right)^4 - 1$	$r_{ef} = \left(1+\frac{r}{12}\right)^{12} - 1$
Semi-annual rate	$r_{ef} = (1+r)^{1/2} - 1$	$r/2$	$r_{ef} = \left(1+\frac{r}{4}\right)^2 - 1$	$r_{ef} = \left(1+\frac{r}{12}\right)^6 - 1$
Quarterly rate	$r_{ef} = (1+r)^{1/4} - 1$	$r_{ef} = \left(1+\frac{r}{2}\right)^{1/2} - 1$	$r/4$	$r_{ef} = \left(1+\frac{r}{12}\right)^3 - 1$
Monthly rate	$r_{ef} = (1+r)^{1/12} - 1$	$r_{ef} = \left(1+\frac{r}{2}\right)^{1/6} - 1$	$r_{ef} = \left(1+\frac{r}{4}\right)^{1/3} - 1$	$r/12$

Average interest rate

$$K_n = K_0\left(1 + n_1 r_1 + n_2 r_2 + \cdots + n_p r_p\right) \qquad n = n_1 + n_2 + \cdots + n_p$$

$$r_{av} = \left(n_1 r_1 + n_2 r_2 + \cdots + n_p r_p\right)/n$$

$$K_n = K_0\left(1 + r_1\right)^{n_1}\left(1 + r_2\right)^{n_2} \cdots \left(1 + r_p\right)^{n_p} \qquad n = n_1 + n_2 + \cdots + n_p$$

$$r_{av} = \sqrt[n]{\left(1 + r_1\right)^{n_1}\left(1 + r_2\right)^{n_2} \cdots \left(1 + r_p\right)^{n_p}} - 1$$

Average interest rate

$$K_n = K_0 (1 - r_1)^{-n_1} (1 - r_2)^{-n_2} \cdots (1 - r_p)^{-n_p} \qquad n = n_1 + n_2 + \cdots + n_p$$

$$r_{av} = 1 - \sqrt[n]{(1 - r_1)^{n_1} (1 - r_2)^{n_2} \cdots (1 - r_p)^{n_p}}$$

$$K_n = K_0 e^{n_1 r_1} e^{n_2 r_2} \cdots e^{n_p r_p} = K_0 e^{n_1 r_1 + n_2 r_2 + \cdots + n_p r_p} \qquad n = n_1 + n_2 + \cdots + n_p$$

$$r_{av} = (n_1 r_1 + n_2 r_2 + \cdots + n_p r_p)/n$$

Example

A company borrowed money from four banks:

- Bank A 1000 PLN, 2 months, simple interest at 18% per annum,
- Bank B 1200 PLN, 4 months, simple interest at 20% per annum,
- Bank C 1100 PLN, 3 months, simple interest at 19% per annum,
- Bank D 1300 PLN , 5 months, simple interest at 21% per annum.

Does the company benefit more if interest rate is the same in all banks and amounts 19.5% p.a.

$$K_n = 1000\left(1 + 2 \cdot \frac{0.18}{12}\right) + 1200\left(1 + 4 \cdot \frac{0.2}{12}\right) + 1100\left(1 + 3 \cdot \frac{0.19}{12}\right) + 1300\left(1 + 5 \cdot \frac{0.21}{12}\right)$$

$$K_n = 1000\left(1 + 2 \cdot \frac{r_{av}}{12}\right) + 1200\left(1 + 4 \cdot \frac{r_{av}}{12}\right) + 1100\left(1 + 3 \cdot \frac{r_{av}}{12}\right) + 1300\left(1 + 5 \cdot \frac{r_{av}}{12}\right)$$

$$r_{av} = 19.95\% > 19.5\%$$