Financial Mathematics

Lecture 4-5

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- **Period-certain annuity (fixed period annuity)** – a type of annuity that guarantees benefit payments for a designated period of time.
- **Perpetuity** – an annuity that has no end (Perpetuities – annuities continuing perpetually).

Period-certain annuity

$$K_N = K(1+r)^N - A_N$$

Annuity-immediate

$$K = \frac{a}{r}\left(1 - \frac{1}{(1+r)^N}\right)$$

$$a = \frac{K \cdot r}{1-(1+r)^{-N}}$$

Annuity due

$$K = \frac{a \cdot (1+r)}{r}\left(1 - \frac{1}{(1+r)^N}\right)$$

$$a = \frac{1}{1+r}\frac{K \cdot r}{\left(1-(1+r)^{-N}\right)}$$

Period-certain annuity

Annuity-immediate

$$N = -\frac{\ln\left(1 - \frac{r \cdot K}{a}\right)}{\ln(1+r)}$$

Annuity due

$$N = -\frac{\ln\left(1 - \frac{r \cdot K}{(1+r) \cdot a}\right)}{\ln(1+r)}$$

Perpetuity

- A perpetuity immediate

$$K = \frac{a}{r}$$

- A perpetuity due

$$K = \frac{a \cdot (1 + r)}{r}$$

$$K = \lim_{N \to \infty} \frac{a}{r}\left(1 - \frac{1}{(1+r)^N}\right) = \frac{a}{r}$$

Example 1 – Annuity-immediate

$N = 4 \qquad K = 15 \qquad r = 4\%$

N	K_{N-1}	$(1+r)K_{N-1}$	a	K_N
1	15.0	15.6	4.1	11.5
2	11.5	11.9	4.1	7.8
3	7.8	8.1	4.1	4.0
4	4.0	4.1	4.1	0

$$a = \frac{K \cdot r}{1-(1+r)^{-N}}$$

$$K_N = (1+r)K_{N-1} - a$$

Example 2– Annuity-due

$N = 4 \qquad K = 15 \qquad r = 4\%$

N	K_{N-1}	$(1+r)K_{N-1}$	a	K_N
1	15.0	15.6	4.0	11.0
2	11.0	11.5	4.0	7.5
3	7.5	7.8	4.0	3.8
4	3.8	4.0	4.0	0

$$a = \frac{1}{1+r}\frac{K \cdot r}{1-(1+r)^{-N}}$$

$$K_N = (1+r)K_{N-1} - a$$

Example 3 – Unknown interest rate

- Capital 100 PLN, annuity 12 PLN, 10 years, Determine interest rate.

a) Annuity-immediate

$$r = 3.46\%$$

$$1 = (1+r)^N \left(1 - \frac{r \cdot K}{a}\right)$$

b) Annuity-due

$$r = 4.304\%$$

$$1 = (1+r)^N \left(1 - \frac{r \cdot K}{(1+r) \cdot a}\right)$$

Example 4 Annuity-immediate

- An investment of 100 PLN is to be used to make payments of 15 PLN at the end of every year for as long as possible. If the fund earns an annual effective rate of interest 1%, find how many regular payments can be made.

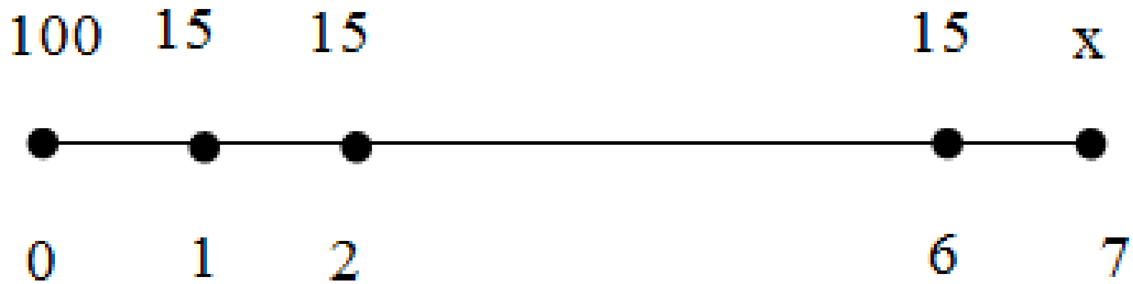
Example 4 Annuity-immediate

$K = 100$ $a = 15$ $r = 0.01$

$$N = -\frac{\ln\left(1 - \frac{r \cdot K}{a}\right)}{\ln(1+r)}$$

$$\boxed{N = 6.93}$$

Example 4 – Additional annuity



$$\left(K(1+r)^6 - A_6\right)\cdot(1+r) = x$$

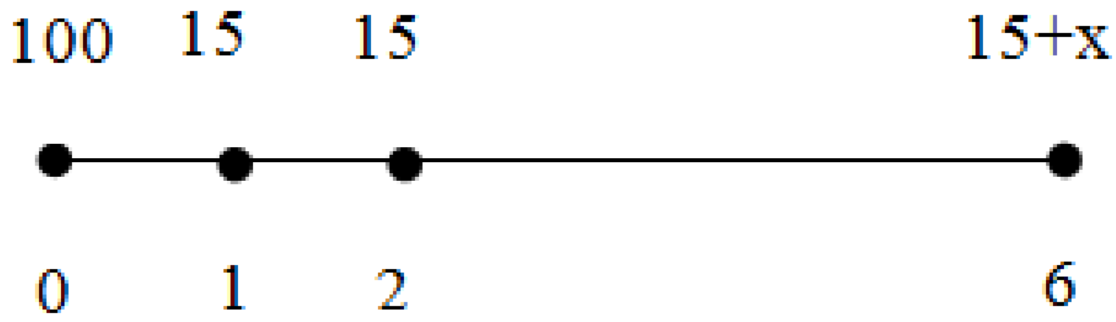
$$A_6 = a\cdot\frac{(1+r)^6 - 1}{r}$$

$$K(1+r)^6 - A_6 = \frac{x}{1+r}$$

$$x = 14.011$$

$$K = \frac{A_6}{(1+r)^6} + \frac{x}{(1+r)^7}$$

Example 4 – Enlargement of one of the payment



$$K(1+r)^6 - A_6 = x$$

$$x = 13.87$$

Example 4 – New annuity

$$a = \frac{K \cdot r}{1-(1+r)^{-7}} \qquad a = \frac{100 \cdot 0.01}{1-(1+0.01)^{-7}} = 14.86$$

Example 5

- A loan of 2400 PLN is to be repaid by 10 equal annual instalments. The rate of interest for the transaction is 10% per annum. Find the amount of each annual repayment, assuming that payments are made:

a) in arrear

b) in advance

$$a = \frac{K \cdot r}{1-(1+r)^{-N}}$$

$$a = \frac{1}{1+r}\frac{K \cdot r}{1-(1+r)^{-N}}$$

$$a = 253.40$$

$$a = 250.89$$