Financial Mathematics

Lecture 6-7

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- **Long-term loans – repayment methods**

- Equal principal payments per time period
- Equal total payments per time period

- Loan amount – the size or value of the loan
- Interest rate – the annual stated rate of the loan
- Number of payments – the total numbers of payments to pay off the given loan amount
- Payment frequency – loans payments are due monthly (quarterly, annually).
- Compounding coincides with payments (Compounding doesn’t coincide with payments)

- **Loan payment = principal payment + interest payment**

- The amortization schedule shows – for each payment – how much of the payment goes toward the loan principal, and how much is paid on interest.

Example 1 – Loan Amortization Schedule

- An investor borrowed 100 PLN. The loan was for four quarters at 20% annual interest (compounding quarterly).

$$S = 100 \qquad N = 4 \qquad r = \frac{0.2}{4} = 0.05$$

Loan amortization schedule – equal principal payments

(interest payment as a percent of the previous principal balance)

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	100	25	5	30	75
2	75	25	3.75	28.75	50
3	50	25	2.5	27.5	25
4	25	25	1.25	26.25	0
Total		**100**	**12.5**	**112.5**	
	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

Loan amortization schedule – equal principal payments
(interest payment as a percent of the repaid loan)

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	100	25	1.25	26.25	75
2	75	25	2.5	27.5	50
3	50	25	3.75	28.75	25
4	25	25	5	30	0
Total		**100**	**12.5**	**112.5**	
	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

Loan amortization schedule – given principal payments

(interest payment as a percent of the previous principal balance)

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	100	10	5	15	90
2	90	20	4.5	24.5	70
3	70	20	3.5	23.5	50
4	50	50	2.5	52.5	0
Total		**100**	**15.5**	**115.5**	

	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

Loan amortization schedule

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	100	0	5	5	100
2	100	0	5	5	100
3	100	0	5	5	100
4	100	100	5	105	0
Total		**100**	**20**	**120**	

	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

Equal total payments

$$S(1+r)^N = A_1(1+r)^{N-1} + A_2(1+r)^{N-2} + \cdots + A_N$$

$$S = \frac{A_1}{1+r} + \frac{A_2}{(1+r)^2} + \cdots + \frac{A_N}{(1+r)^N}$$

Periodic payment

$$S(1+r)^N = A\frac{(1+r)^N - 1}{r}$$

$$\boxed{A = \frac{S \cdot r \cdot (1+r)^N}{(1+r)^N - 1}}$$

Equal total payments

$$Z_n = r \cdot S_{n-1} \qquad T_n = S_{n-1} - S_n \qquad A_n = T_n + Z_n$$

$$S_n = S(1+r)^n - \left(A_1(1+r)^{n-1} + A_2(1+r)^{n-2} + \cdots + A_{n-1}(1+r)\right) - A_n$$

$$S_n = (1+r)\left(S(1+r)^{n-1} - \left(A_1(1+r)^{n-2} + A_2(1+r)^{n-3} + \cdots + A_{n-1}\right)\right) - A_n$$

$$S_n = (1+r)S_{n-1} - A_n$$

Loan amortization schedule – equal total payments

(interest payment as a percent of the previous principal balance)

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	100	23.2	5	28.2	76.8
2	76.8	24.36	3.84	28.2	52.44
3	52.44	25.58	2.62	28.2	26.86
4	26.86	26.86	1.34	28.2	0
Total		**100**	**12.8**	**112.8**	
	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

Loan amortization schedule – given total payments

(interest payment as a percent of the previous principal balance)

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	100	15	5	20	85
2	85	25.75	4.25	30	59.25
3	59.25	37.04	2.96	40	22.21
4	22.21	22.21	1.11	23.32	0
Total		**100**	**13.32**	**113.32**	
	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

$$S(1+r)^4 = A_1(1+r)^3 + A_2(1+r)^2 + A_3(1+r) + \boxed{A_4}$$

Equal total payments

(continuously compounded interest)

$$Se^{r\cdot N} = A_1 e^{r(N-1)} + A_2 e^{r(N-2)} + \cdots + A_N$$

$$Se^{r\cdot N} = A\frac{e^{r\cdot N}-1}{e^r-1}$$

$$\boxed{A = S\cdot e^{r\cdot N}\cdot\frac{e^r-1}{e^{r\cdot N}-1}}$$

$$\boxed{Z_n = S_{n-1}\cdot\left(e^r-1\right) \qquad T_n = S_{n-1}-S_n \qquad A_n = T_n + Z_n}$$

Loan amortization schedule – equal total payments

(continuously compounded interest)

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	100	23.16	5.13	28.28	76.84
2	76.84	24.34	3.94	28.28	52.5
3	52.5	25.59	2.69	28.28	26.91
4	26.91	26.91	1.38	28.28	0
Total		**100**	**13.14**	**113.14**	

	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

Example – Debt consolidation loans

- 12 monthly payments of 10 PLN, 15% annual interest rate (compounding quarterly)
- 5 semi-annual payments of 100 PLN, 12% annual interest rate (compounding monthly).

- 10 quarterly payments of consolidated loan, 18% annual interest rate (compounding annually)

Example – Debt consolidation loans

$$S = \frac{A}{(1+r)^N} \frac{(1+r)^N - 1}{r}$$

$$S_1 = \frac{10}{(1+r_1)^{12}} \frac{(1+r_1)^{12} - 1}{r_1} \qquad r_1 = \left(1 + \frac{0.15}{4}\right)^{\frac{1}{3}} = 0.01235$$

$$S_2 = \frac{100}{(1+r_2)^5} \frac{(1+r_2)^5 - 1}{r_2} \qquad r_2 = \left(1 + \frac{0.12}{12}\right)^6 = 0.06152$$

$$S_1 = 110.9 \qquad S_2 = 419.5$$

Example – Debt consolidation loans

$$S = 530.4$$

$$A = \frac{S \cdot r \cdot (1+r)^N}{(1+r)^N - 1}$$

$$r = (1 + 0.18)^{\frac{1}{4}} = 0.04225$$

$$A = 66.1$$