Financial Mathematics

Lecture 8

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Example 1a – Loan Amortization Schedule

- An investor borrowed 1000 PLN. The loan was for 6 months at 24% annual interest (compound interest rate).
- Create a loan amortization schedule if since the fourth month the annual interest is 18%.

$$S = 1000 \qquad N = 6 \qquad r = \frac{0.24}{12} = 0.02$$

$$A = \frac{S \cdot r \cdot (1+r)^N}{(1+r)^N - 1} \qquad A = \frac{1000 \cdot 0.02 \cdot (1+0.02)^6}{(1+0.02)^6 - 1} = 178.5$$

$$S_3 = 514.8 \qquad N = 3 \qquad r = \frac{0.18}{12} = 0.015$$

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	1000	158.5	20.0	178.5	841.5
2	841.5	161.7	16.8	178.5	679.8
3	679.8	164.9	13.6	178.5	514.8
4	514.8	169.1	7.7	176.8	345.8
5	345.8	171.6	5.2	176.8	174.2
6	174.2	174.2	2.6	176.8	0
Total		**1000**	**65.9**	**1065.9**	
	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

Example 1b – Loan Amortization Schedule

- An investor borrowed 1000 PLN. The loan was for 6 months at 24% annual interest (compound interest rate).
- Create a loan amortization schedule if the investor pays additional 100 PLN with the third payment.

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	1000	158.5	20.0	178.5	841.5
2	841.5	161.7	16.8	178.5	679.8
3	679.8	264.9	13.6	278.5	414.8
4	414.8	170.2	8.3	178.5	244.6
5	244.6	173.6	4.9	178.5	71.0
6	71.0	71.0	1.4	72.4	0.0
Total		**1000**	**65.0**	**1065.0**	
	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

Example 1c – Loan Amortization Schedule

- An investor borrowed 1000 PLN. The loan was for 6 months at 24% annual interest (compound interest rate).
- Create a loan amortization schedule if the investor doesn’t pay the fourth payment. He pays it plus interest with the fifth payment.

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	1000	158.5	20.0	178.5	841.5
2	841.5	161.7	16.8	178.5	679.8
3	679.8	164.9	13.6	178.5	514.8
4	514.8	-10.3	10.3	0	525.1
5	525.1	350.1	10.5	360.6	175.0
6	175.0	175.0	3.5	178.5	0
Total		**1000**	**74.7**	**1074.7**	
	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

Example 1d – Loan Amortization Schedule

- An investor borrowed 1000 PLN. The loan was for 6 months at 24% annual interest (compound interest rate).
- Create a loan amortization schedule if the first payment is postponed for two months.

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	1040.4	164.9	20.8	185.7	875.5
2	875.5	168.2	17.5	185.7	707.2
3	707.2	171.6	14.1	185.7	535.6
4	535.6	175.0	10.7	185.7	360.6
5	360.6	178.5	7.2	185.7	182.1
6	182.1	182.1	3.6	185.7	0
Total		**1040.4**	**74.0**	**1114.4**	
	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

Example 1e – Loan Amortization Schedule

- An investor borrowed 1000 PLN. The loan was for 6 months at 24% annual interest (compound interest rate).
- Create a loan amortization schedule if the investor pays two payments, than he doesn’t pay for 3 months. The investor begins to pay off the loan again in the sixth month paying three equal payments every two months. Since the third month the annual interest rate is 18%.

n	S_{n-1}	T_n	Z_n	A_n	S_n
1	1000	158.5	20.0	178.5	841.5
2	841.5	161.7	16.8	178.5	679.8
6	710.8	237.0	10.7	247.7	473.8
8	480.9	240.5	7.2	247.7	240.4
10	244.0	244.0	3.7	247.7	0
Total		**1000**	**58.4**	**1058.4**	
	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance

$$S_5 = 679.8 \cdot (1.015)^3 = 710.8$$

$$S_5 = \frac{A_6}{1+r} + \frac{A_8}{(1+r)^3} + \frac{A_{10}}{(1+r)^5}$$

$$A_6 = A_8 = A_{10} = A$$

$$710.8 = \frac{A}{1.015} + \frac{A}{(1.015)^3} + \frac{A}{(1.015)^5}$$

$$S_7 = 473.8 \cdot 1.015 = 480.9$$

Example 2

- An investor borrowed 50 PLN. Find how many payments of 15 PLN should be made if the effective rate of interest is 10%.
- Solve the problem of non-integer number of payments.

$$S(1+r)^N = A\frac{(1+r)^N - 1}{r}$$

$$S = 50 \qquad A = 15$$

$$r = 0.1$$

$$N = \frac{\ln 1.5}{\ln 1.1} = 4.25$$

	Previous principal balance	Principal payment	Interest payment	Total payment	Principal balance
n	S_{n-1}	T_n	Z_n	A_n	S_n
1	50	10	5	15	40
2	40	11	4	15	29
3	29	12.1	2.9	15	16.9
4	16.9	13.3	1.7	15	**3.59**
	3.59	3.59	0.36	3.95	

Additional payment

Enlargement of one of the payment

$$A_1 = A_2 = A_3 = 15 \qquad A_4 = 18.59$$

$$A_2 = A_3 = A_4 = 15 \qquad A_1 = 17.70$$

$$A_1 = A_3 = A_4 = 15 \qquad A_2 = 17.97$$

$$A_1 = A_2 = A_4 = 15 \qquad A_3 = 18.26$$

New payments

$$N = 4$$

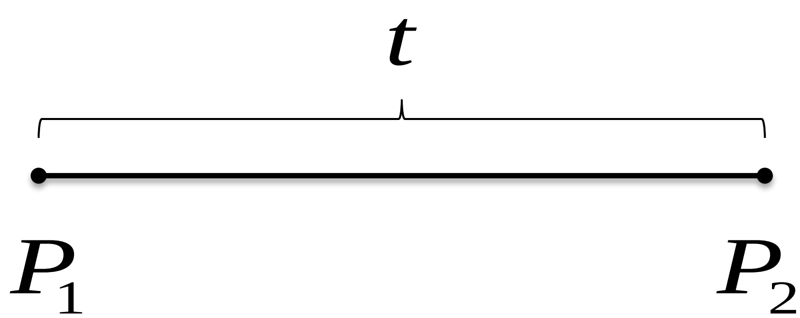
$$A = \frac{S \cdot r \cdot (1+r)^N}{(1+r)^N - 1}$$

$$A = 15.77$$

Treasury bills

- Treasury bills are discounted short-term debt securities with maturities of up to one year.
- Treasury bills are sold at a discount off their nominal value.
- Treasury bills represent an important instrument of governmental fiscal policy and the central bank's monetary policy.
- The nominal value is payable to the final holder upon redemption on maturity.
- Nominal/face value – 10 000 PLN in Poland.
- Maturity – the date the bill is redeemed and the investor is paid the face value amount.
- Regular Treasury bill series are issued weekly (13, 26 or 52 weeks in Poland).

Bill valuation methods

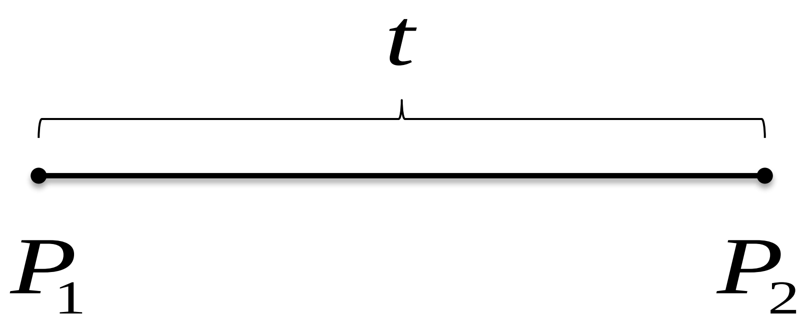


- P_1 – purchase price (at which investor can buy)
- P_2 – nominal/face value (principal)
- t – number o days from purchase to maturity

Bill valuation methods

- The method applied to determine the value of bills depends on whether the bill price is based on the rate of return (r) or the rate of discount (d).
- Bond prices are quoted relative to a 100 PLN face/nominal value.

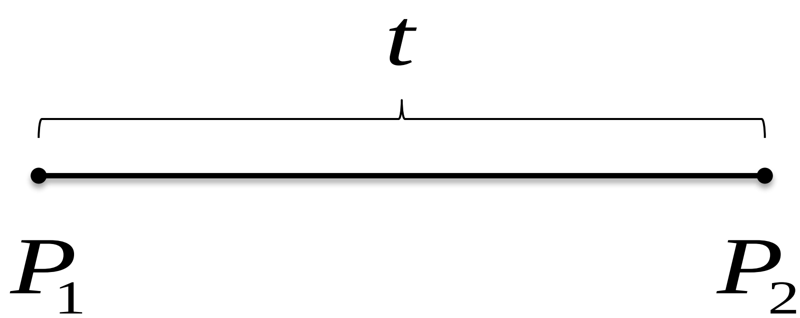
Treasury bills – the rate of return



$$\frac{P_2 - P_1}{P_1} \quad \rightarrow \quad t$$

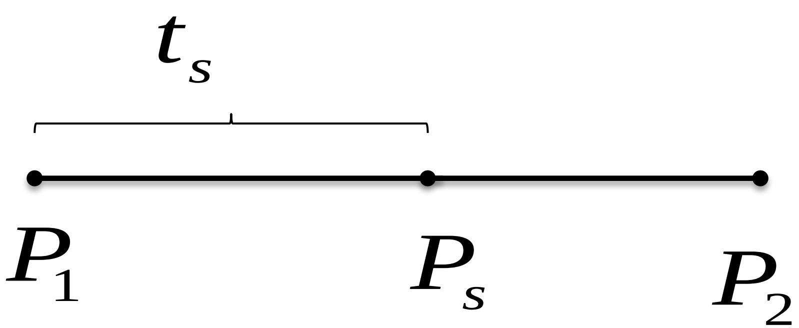
$$r \quad \rightarrow \quad 360$$

Treasury bills – the rate of return



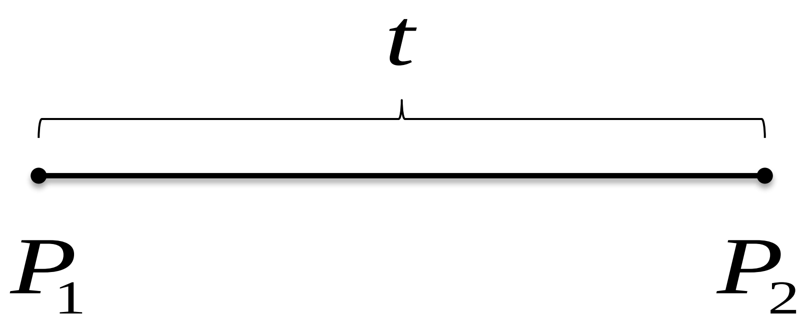
$$r = \frac{P_2 - P_1}{P_1} \cdot \frac{360}{t}$$

Treasury bills – the rate of return for the holding period



$$r_s = \frac{P_s - P_1}{P_1} \cdot \frac{360}{t_s}$$

Treasury bills – the discount rate



$$d = \frac{P_2 - P_1}{P_2} \cdot \frac{360}{t}$$

Treasury bills – price of the Treasury bills

- The price per 100 PLN principal (bills quoted on the basis of the rate of return).

$$P = \frac{360}{r \cdot t + 360} \cdot 100$$

- The price per 100 PLN principal (bills quoted on the basis of the discount rate)

$$P = \left(1 - \frac{d \cdot t}{360}\right) \cdot 100$$

Treasury bills

$$\frac{360}{r \cdot t + 360} \cdot 100 = \left(1 - \frac{d \cdot t}{360}\right) \cdot 100$$

$$r = \frac{d}{1 - d \cdot \frac{t}{360}}$$

$$d = \frac{r}{1 + r \cdot \frac{t}{360}}$$

The rate of return for the known discount rate

The discount rate for the known rate of return

Example 1 – Treasury bills

Investor buys Treasury bills at the primary market with maturity 26 weeks. The nominal value of bills is 1.5 million PLN. The investors pays 97.9005 per a 100 PLN.

$$9790.05 \cdot 150 = 1468508$$

- The rate of return

$$r = \frac{100 - 97.9005}{97.9005} \cdot \frac{360}{182} = 0.04242$$

- The discount rate

$$d = \frac{100 - 97.9005}{100} \cdot \frac{360}{182} = 0.04153$$

Example 2 – Treasury bills

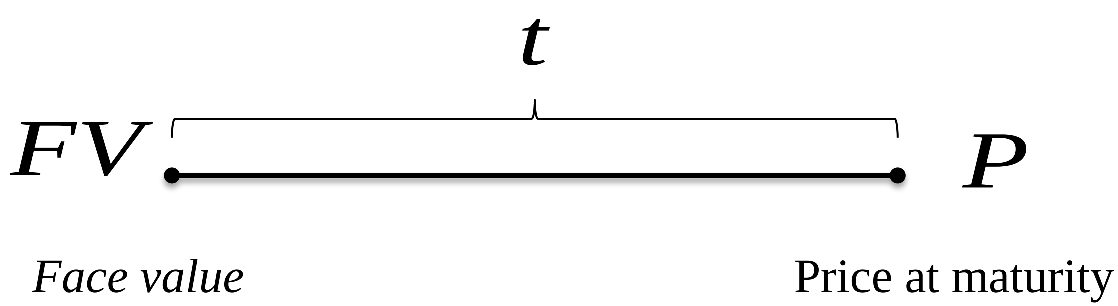
- Assuming that the Treasury bills have been issued at a rate of return of 9% per 60 days, calculate the appropriate discount rate.

$$d = \frac{r}{1 + r \cdot \frac{t}{360}} = \frac{0.09}{1 + 0.09 \cdot \frac{60}{360}} = 0.08867$$

A certificate of deposit – CD

- A certificate of deposit is a savings certificate with a fixed maturity date, specified fixed interest rate issued by commercial banks.
- A CD restricts access to the funds until the maturity date of the investment.

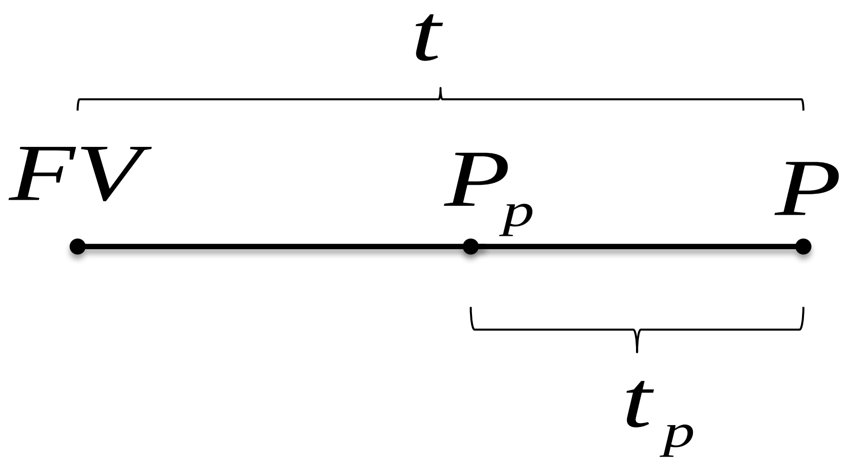
A certificate of deposit



$$P = FV \cdot \left(1 + r_k \cdot \frac{t}{360}\right)$$

r_k – interest rate

A certificate of deposit



Number of days
from purchase to maturity

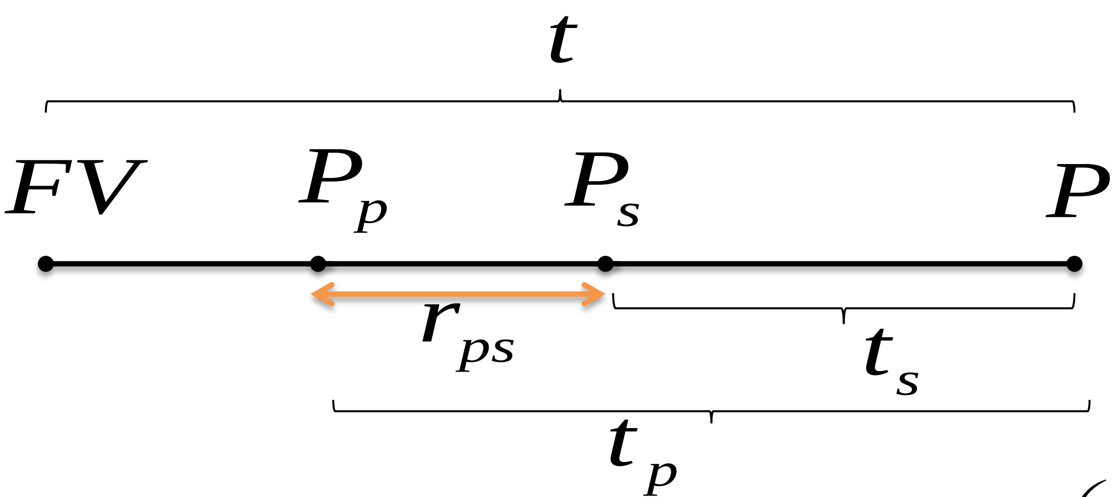
$$FV \cdot \left(1 + r_k \cdot \frac{t}{360}\right) = P_p \cdot \left(1 + r_p \cdot \frac{t_p}{360}\right)$$

$$P_p = \frac{FV \cdot \left(1 + r_k \cdot \frac{t}{360}\right)}{\left(1 + r_p \cdot \frac{t_p}{360}\right)}$$

$$\boxed{P_p = \frac{100 \cdot \left(1 + r_k \cdot \frac{t}{360}\right)}{\left(1 + r_p \cdot \frac{t_p}{360}\right)}}$$

Purchase price

CD – the rate of return for the holding period



$$r_{ps} = \frac{P_s - P_p}{P_p} \cdot \frac{360}{t_p - t_s}$$

$$r_{ps} = \left(\frac{1 + r_p \cdot \frac{t_p}{360}}{1 + r_s \cdot \frac{t_s}{360}} - 1 \right) \cdot \frac{360}{t_p - t_s}$$

Purchase price

$$P_p = \frac{100 \cdot \left(1 + r_k \cdot \frac{t}{360}\right)}{\left(1 + r_p \cdot \frac{t_p}{360}\right)}$$

$$P_s = \frac{100 \cdot \left(1 + r_k \cdot \frac{t}{360}\right)}{\left(1 + r_s \cdot \frac{t_s}{360}\right)}$$

Sell price

Example 3 – CD

- Investor buys CD at the primary market with maturity 13 weeks. The nominal value of CD is 1 million PLN. The rate of return is 20%.
- Calculate the price at maturity

91 days

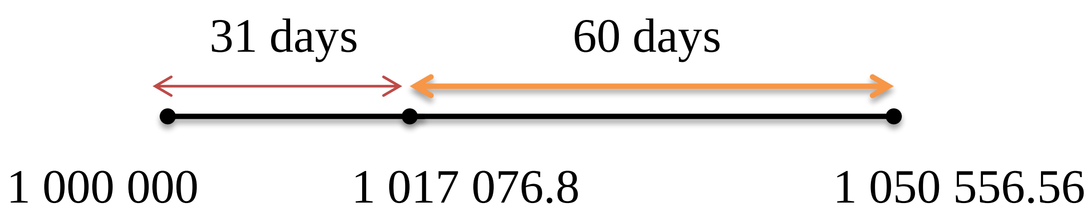
1 000 000 $\quad r_k = 0.2 \quad$ 1 050 556.56

100 PLN $\quad$ 105.0556 PLN

$$P = 1000000 \cdot \left(1 + 0.2 \cdot \frac{91}{360}\right) = 1050556.556$$

Example 3 – CD

- After 31 days the investor sells CD at a 19.75% rate of return.



$$P_s = \frac{1000000 \cdot \left(1 + 0.2 \cdot \frac{91}{360}\right)}{\left(1 + 0.1975 \cdot \frac{60}{360}\right)} = 1017076.8$$

101.7077 – dirty price

101.7077-1.7222 = 99.9855 – clean price

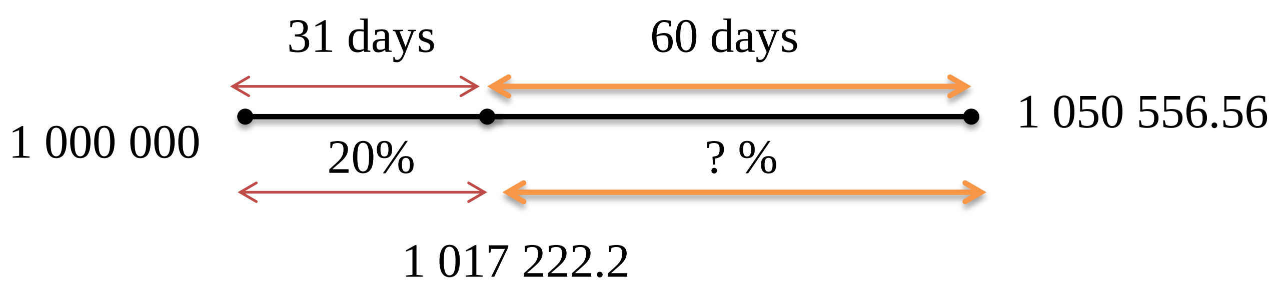
Interest for 100 PLN

$$100 \cdot \frac{0.2 \cdot 31}{360} = 1.722$$

1 017 076.8-17 222.2= 999 854.6

-145,4 PLN

Example 3 – CD



$$P_s = 1000000 \cdot \left(1 + 0.2 \cdot \frac{31}{360}\right) = 1017222.2$$

$$r_s = \frac{1050556.56 - 1017222.2}{1017222.2} \cdot \frac{360}{60} = \boxed{0.1966}$$