Financial Mathematics

Lecture 9-10

Dr Wioletta Nowak

Fundamentals of bond valuation

- Bond – a loan between a borrower (issuer) and a lender (investor, creditor)
- The issuer promises to make regular interest payments to the investor at a specified rate (the **coupon rate**) on the amount it has borrowed (the **face/par amount**) until a specified date (the **maturity date**).
- Once the bond matures, the interest payments stop and the issuer is required to repay the face amount of the principal to the investor.

Fundamentals of bond valuation

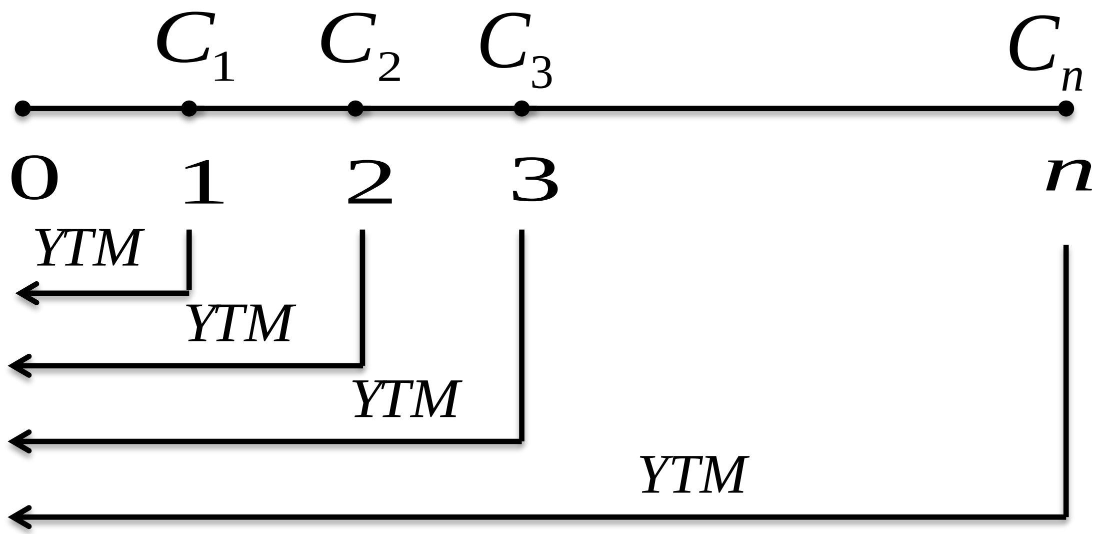
- Bonds can be priced at **a premium**, **discount**, or **at par**.
- If the bond's price is higher than its par value, it will sell at a premium because its interest rate is higher than current prevailing rates.
- If the bond's price is lower than its par value, the bond will sell at a discount because its interest rate is lower than current prevailing interest rates.

Fundamentals of bond valuation

- Bond valuation is the determination of the fair price of a bond.
- The price of bond is the sum of the present values of all expected coupon payments plus the present value of the par value at maturity.
- Yield to maturity – is the internal rate of return earned by investor who buys the bond today at the market price, assuming that the bond will be held until maturity.

Bond pricing – coupon bonds

- C_i – income from the ownership bonds at time i, n – number of payments, YTM – yield to maturity, P – bond price



$$P = \frac{C_1}{1+YTM} + \frac{C_2}{(1+YTM)^2} + \cdots + \frac{C_n}{(1+YTM)^n} = \sum_{i=1}^{n} \frac{C_i}{(1+YTM)^i}$$

Bond pricing – coupon bonds

- **Constant coupon rate**, C – coupon payment, M – value at maturity or par value, n – number of payments, YTM – yield to maturity, P – bond price

$$P = \frac{C}{1+YTM} + \frac{C}{(1+YTM)^2} + \cdots + \frac{C+M}{(1+YTM)^n}$$

$$P = \frac{C}{1+YTM}\left(1 + \frac{1}{1+YTM} + \cdots + \frac{1}{(1+YTM)^{n-1}}\right) + \frac{M}{(1+YTM)^n}$$

$$P = C \cdot \frac{1-(1+YTM)^{-n}}{YTM} + \frac{M}{(1+YTM)^n}$$

Example 1

Suppose a 4-year bond with the value at maturity of 100 PLN and a coupon rate of 10%.

Time to maturity	**Price of bond**			**Premium**	**Discount**	**Percent of premium decline**	**Percent of discount decline**
	YTM= 9%	YTM=10%	YTM=11%				
4	103.24	100	96.90	3.24	3.10	–	–
3	102.53	100	97.56	2.53	2.44	21.87%	21.23%
2	101.76	100	98.29	1.76	1.71	30.51%	29.92%
1	100.92	100	99.10	0.92	0.9	47.85%	47.39%

$$\frac{3.24-2.53}{3.24}=0.2187$$

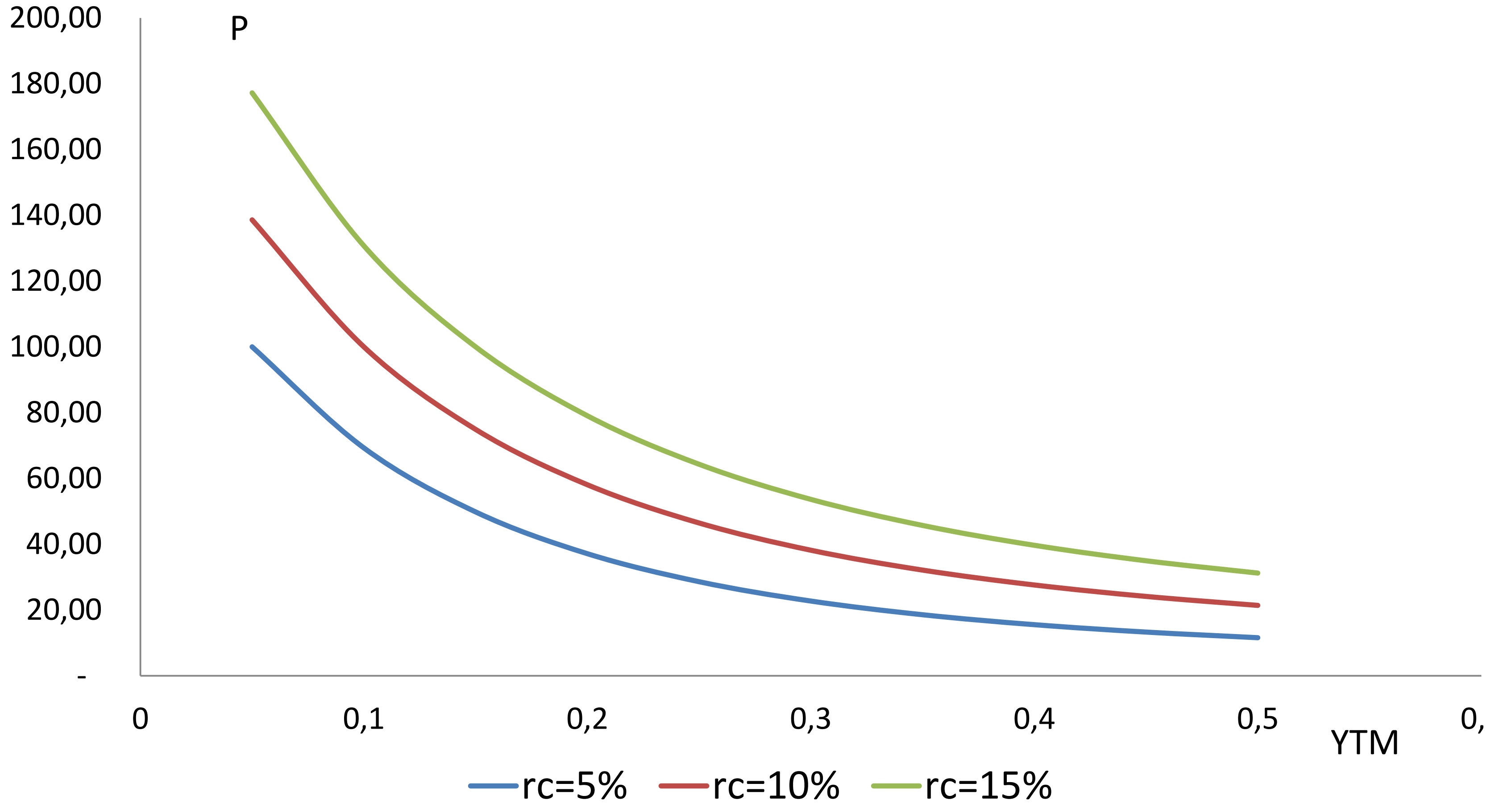
Example 2

- Suppose a 3-year bond with the value at maturity of 100 PLN.

Coupon rate	**Price of bond**		**Percent of decrease**
	YTM = 8%	YTM = 12%	
10%	105.15	95.20	9.47%
15%	118.04	107.21	9.18%

$$\frac{105.15 - 95.2}{105.15} = 0.0947$$

# Example 3
n=10 M=100

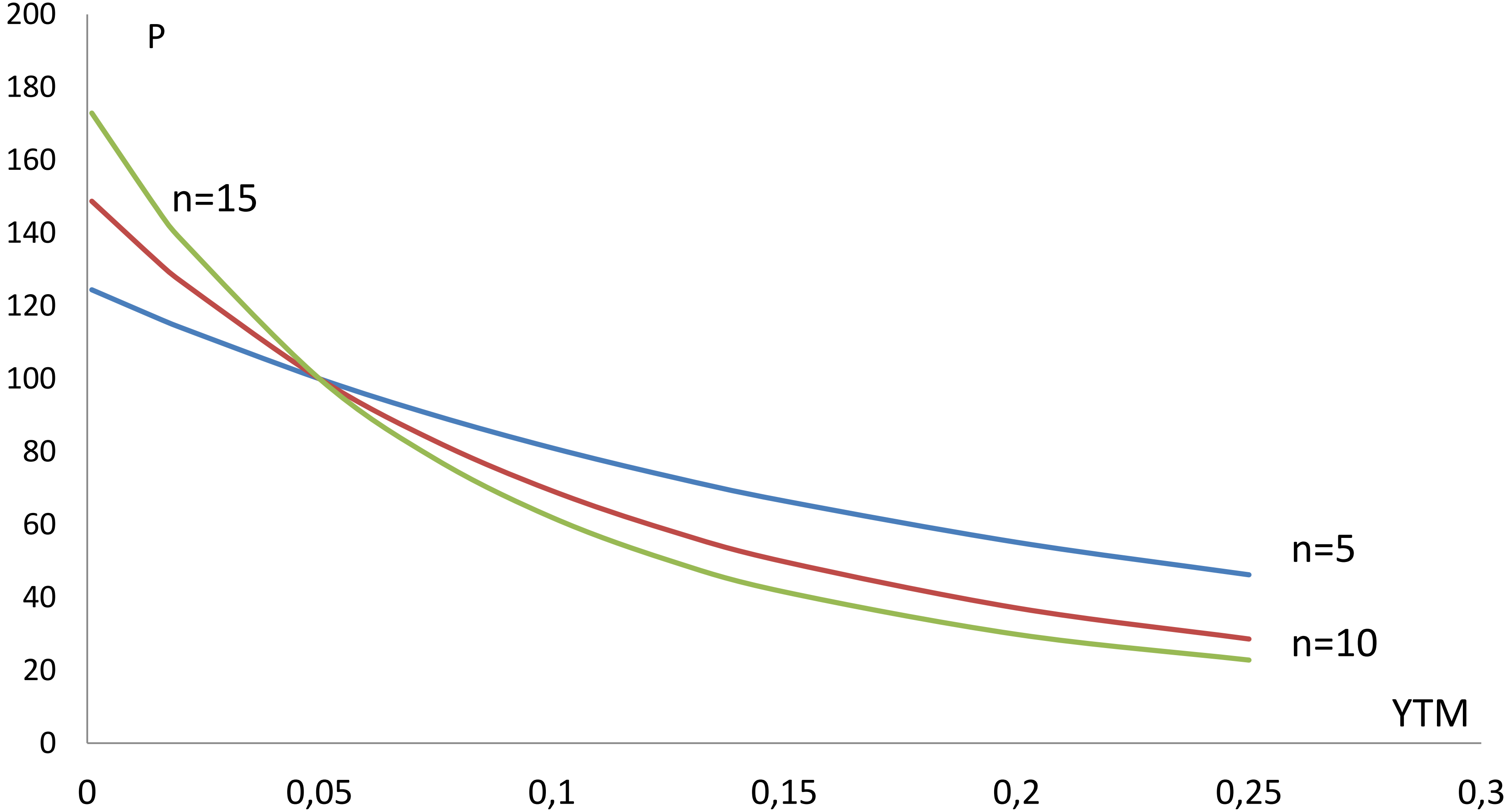


Example 4

- Suppose a bond with the value at maturity of 100 PLN and a coupon rate of 10%.

Time to maturity (in years)	**Price of bond**		**Percent of decrease**
	YTM = 8%	YTM = 12%	
3	105.15	95.20	9.47%
5	107.99	92.79	14.07%

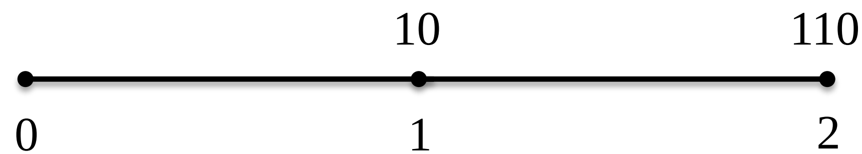
# Example 5
M=100, rc=5%



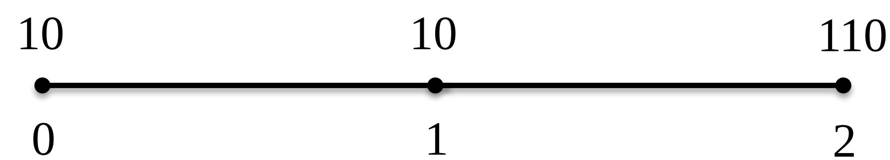
Example 6

- Calculate the price of a bond with a par value of 100 PLN to be paid in two years (after and before the coupon payment), a coupon rate of 10%, and a required yield of 9%.

$$P = \frac{10}{1.09} + \frac{110}{(1.09)^2} = 101.76$$



$$P = 10 + \frac{10}{1.09} + \frac{110}{(1.09)^2} = 111.76$$



Example 7

- Calculate the price of a bond with a par value of 100 PLN to be paid in two years and six months, a coupon rate of 10%, and a required yield of 8%. An annual coupon payment.

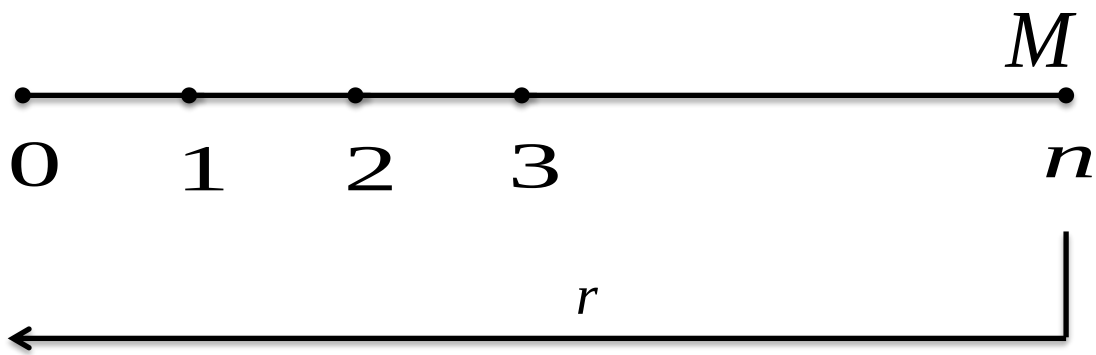
$$P = \frac{10}{(1.08)^{0.5}} + \frac{10}{(1.08)^{1.5}} + \frac{110}{(1.08)^{2.5}} = 109.28$$

Zero-coupon bonds

- Zero-coupon or accrual bonds do not pay a coupon. Instead, these types of bonds are issued at a deep discount and pay the full face value at maturity.

Fundamentals of bond valuation – bond price

- **Zero-coupon bond**, M – value at maturity, n – number of periods, r – interest rate, P – bond price



$$P = \frac{M}{(1+r)^n}$$

Example 8

- Calculate the price of a zero-coupon bond that is maturing in one and a half years, has a par value of 100 PLN and a required yield of 5%.

$$P = \frac{100}{(1+0.05)^{1.5}} = 92.94$$

Perpetual bond – pricing

- A bond with no maturity date. Issuers pay coupons forever.

$$P = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \quad \cdots$$

$$P = \frac{C}{r}$$

- C – coupon interest on bond, r – an expected yield for maximum term available

Duration of a financial asset that consists of fixed cash flows

- The weighted average of the times until the fixed flows are received

$$D = w_1 t_1 + w_2 t_2 + \cdots + w_n t_n$$

$$w_i = \frac{PV_i}{PV} \qquad PV = PV_1 + PV_2 + \cdots + PV_n$$

PV_i – the present value of the payment at time t_i

The Macaulay duration

$$D = \frac{\sum_{k=1}^{n} \frac{k \cdot C_k}{(1+YTM)^k}}{\sum_{k=1}^{n} \frac{C_k}{(1+YTM)^k}}$$

$$D = \frac{\sum_{k=1}^{n} \frac{k \cdot C_k}{(1+YTM)^k}}{P}$$

k – period in which a coupon is received

The Macaulay duration

- The weighted average of the time of receipt of a bond's fixed cash flow payments.
- The balance point of a group of cash flows.
- It helps to compare bonds with different time to maturity and different coupon rates.
- The higher a bond's coupon – the shorter the Macaulay duration.
- The longer a bond's maturity the greater its duration.

$n \to \infty$ $\quad D \to \dfrac{1}{YTM} + \dfrac{1}{m}$ $\quad$ m – a frequency of coupon

The Macaulay duration

- The higher the YTM – the shorter the Macaulay duration
- Higher frequency of coupon payment – the shorter the Macaulay duration.
- Zero-coupon bonds have durations equal to their maturities.

Example 9

- Suppose a 3-year bond with a value at maturity of 100 PLN, coupon rate of 5%, YTM of 10%. What is the Macaulay duration of the bond?

$$D = \frac{\frac{1 \cdot 5}{1.1} + \frac{2 \cdot 5}{(1.1)^2} + \frac{3 \cdot 105}{(1.1)^3}}{\frac{5}{1.1} + \frac{5}{(1.1)^2} + \frac{105}{(1.1)^3}} = 2.849$$

The modified Macaulay duration

- The modified Macaulay duration measures the price sensitivity of a bond when there is a change in the yield to maturity

$$\frac{\Delta P}{P} = -D \cdot \frac{\Delta YTM}{1+YTM}$$

$$\frac{\Delta P}{P} = -D_M \cdot \Delta YTM$$

$$D_M = \frac{D}{1+YTM}$$

$$D_M = \frac{D}{1+YTM/m}$$

m – a frequency of coupon

Example 10

- Suppose a 3-year bond with the value at maturity of 100 PLN, a coupon rate of 5% and the YTM of 10%. How much will the bond price change if the YTM increases by 1 percentage point (decreases by 1 percentage point).

$$D = 2.849 \qquad D_M = \frac{2.849}{1.1} = 2.59 \qquad \frac{\Delta P}{P} = -D_M \cdot \Delta YTM$$

$$\Delta YTM = +1\% \qquad \Delta YTM = -1\%$$

$$\frac{\Delta P}{P} = -2.59\% \qquad \frac{\Delta P}{P} = +2.59\%$$

Share evaluation models

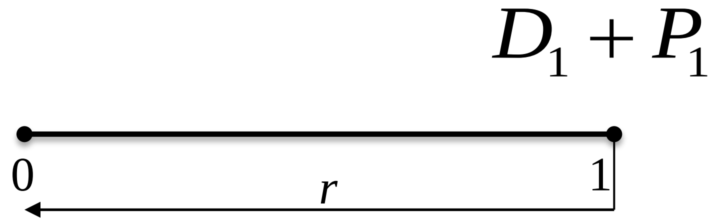
- **Dividend discount model** – method of estimating the value of a share stock as the present value of all expected future dividend payments.

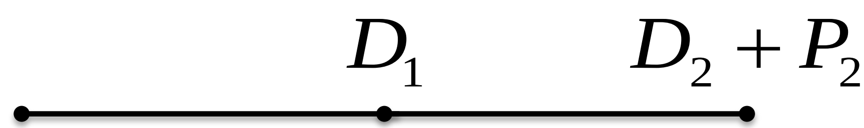
- **Constant dividend model**
- **Constant dividend growth rate model** – Gordon model

- **Two-stage dividend growth model**
- **Multistage dividend growth model**

Dividend discount model

$$P = \frac{D_1}{1+r} + \frac{P_1}{1+r}$$





$$P_1 = \frac{D_2}{1+r} + \frac{P_2}{1+r} \quad \Rightarrow \quad P = \frac{D_1}{1+r} + \frac{D_2}{(1+r)^2} + \frac{P_2}{(1+r)^2}$$

Dividend discount model

$$P = \frac{D_1}{1+r} + \frac{D_2}{(1+r)^2} + \frac{P_2}{(1+r)^2}$$

$$P_2 = \frac{D_3}{1+r} + \frac{P_3}{1+r}$$

$$P = \frac{D_1}{1+r} + \frac{D_2}{(1+r)^2} + \frac{D_3}{(1+r)^3} + \frac{P_3}{(1+r)^3}$$

$$P = \sum_{k=1}^{n} \frac{D_k}{(1+r)^k} + \frac{P_n}{(1+r)^n}$$

$$n \to \infty$$

$$P = \sum_{k=1}^{\infty} \frac{D_k}{(1+r)^k}$$

Constant dividend model

$$P = \sum_{k=1}^{\infty} \frac{D_k}{(1+r)^k}$$

$$P = D \cdot \sum_{k=1}^{\infty} \frac{1}{(1+r)^k}$$

$$P = D \cdot \frac{\frac{1}{1+r}}{1-\frac{1}{1+r}} = \frac{D}{r}$$

$$P = \frac{D}{r}$$

Constant dividend growth rate model

- Dividend will grow at a constant growth rate *g*.

$$D_{k+1} = (1+g) \cdot D_k$$

- For a known D_1

$$P = \frac{D_1}{1+r} + \frac{D_1 \cdot (1+g)}{(1+r)^2} + \frac{D_1 \cdot (1+g)^2}{(1+r)^3} + \cdots$$

$$P = D_1 \cdot \sum_{k=1} \frac{(1+g)^{k-1}}{(1+r)^k}$$

Constant perpetual growth model

- Model in which dividends grow forever at a constant rate g, and the growth rate g is strictly less than the discount rate r.

$$g < r$$

Constant perpetual growth model

$$P = D_1 \cdot \sum_{k=1} \frac{(1+g)^{k-1}}{(1+r)^k}$$

$$\sum_{k=1} \frac{(1+g)^{k-1}}{(1+r)^k} = \frac{1}{1+r} \cdot \frac{1}{1-\frac{1+g}{1+r}} = \frac{1}{r-g} \qquad r > g$$

$$P = \frac{D_1}{r-g}$$

$$D_1 = (1+g) \cdot D$$

$$P = \frac{D \cdot (1+g)}{r-g}$$

Example 11

- Suppose the current dividend is 100 PLN. If the discount rate is 10%, what is the value of the stock?

- **Constant dividend discount model** $P = \frac{D}{r} = \frac{100}{0.1} = 1000$

- **Constant perpetual growth model** (suppose dividends are projected to grow at 8% forever)

$$P = \frac{D \cdot (1+g)}{r-g} = \frac{100 \cdot (1+0.08)}{0.1-0.08} = \frac{108}{0.02} = 5400$$

Two-stage dividend growth model

- Dividend grow at a rate g_1 during a first stage of growth lasting n years and thereafter grow at a rate g_2 during a perpetual second stage of growth

$$(g_1 > g_2)$$

$$P = \frac{D_1}{1+r} + \cdots + \frac{D_n}{(1+r)^n} + \frac{P_n}{(1+r)^n}$$

$$D_n = (1+g_1) \cdot D_{n-1}$$

$$D_{n+1} = (1+g_2) \cdot D_n$$

$$P_n = \frac{D_{n+1}}{r-g_2} = \frac{D_n \cdot (1+g_2)}{r-g_2}$$

Example 12

- Suppose a firm has a current dividend of 100 PLN which is expected to grow at the rate of 8% for 3 years, and thereafter grow at the rate of 3%. With a discount rate of 10%, what is the value of stock?

$$P = \frac{D_1}{1+r} + \frac{D_2}{(1+r)^2} + \frac{D_3}{(1+r)^3} + \frac{P_3}{(1+r)^3}$$

$$D_1 = D \cdot (1+g_1)$$

$$D_2 = D \cdot (1+g_1)^2$$

$$D_3 = D \cdot (1+g_1)^3$$

$$P_3 = \frac{D_4}{r-g_2} = \frac{D_3 \cdot (1+g_2)}{r-g_2}$$

$$P_3 = \frac{D \cdot (1+g_1)^3 \cdot (1+g_2)}{r-g_2}$$

Example 12

$$D_1 = 100 \cdot (1 + 0.08) = 108$$

$$D_2 = 108 \cdot 1.08 = 116.64$$

$$D_3 = 116.64 \cdot 1.08 = 125.97$$

$$D_4 = 125.97 \cdot 1.03 = 129.75$$

$$P_3 = \frac{129.75}{0.1 - 0.03} = 1853.58$$

$$P = \frac{108}{1.1} + \frac{116.64}{(1.1)^2} + \frac{125.97}{(1.1)^3} + \frac{1853.58}{(1.1)^3} = 1681.84$$

Example 13

- Dividend is expected to grow at g_1 for 4 years, at g_2 for 2 years, at g_3 for 3 years, and thereafter at g_4

$$P = \frac{D_1}{1+r} + \cdots + \frac{D_9}{(1+r)^9} + \frac{P_9}{(1+r)^9}$$

$$D_1 = D \cdot (1+g_1)$$
$$D_2 = D_1 \cdot (1+g_1)$$
$$D_3 = D_2 \cdot (1+g_1)$$
$$D_4 = D_3 \cdot (1+g_1)$$

$$D_5 = D_4 \cdot (1+g_2)$$
$$D_6 = D_5 \cdot (1+g_2)$$

$$P_9 = \frac{D_{10}}{r-g_4} = \frac{D_9 \cdot (1+g_4)}{r-g_4}$$

$$D_7 = D_6 \cdot (1+g_3)$$
$$D_8 = D_7 \cdot (1+g_3)$$
$$D_9 = D_8 \cdot (1+g_3)$$