Mathematical Economics

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Lecture 2

- The Utility Function,
- Examples of Utility Functions: Normal Good, Perfect Substitutes, Perfect Complements,
- The Quasilinear and Homothetic Utility Functions,
- The Marginal Utility and The Marginal Rate of Substitution,
- The Optimal Choice,
- The Utility Maximization Problem,
- The Lagrange Method

The Utility Function

- A utility is a measure of the relative satisfaction from consumption of various goods.

- A utility function is a way of assigning a number to every possible consumption bundle such that more-preferred bundles get assigned larger numbers then less-preferred bundles.

The Utility Function

- The numerical magnitudes of utility levels have no intrinsic meaning – the only property of a utility assignment that is important is how it orders the bundles of goods.

- The magnitude of the utility function is only important insofar as it ranks the different consumption bundles.

- **Ordinal utility** - consumer assigns a higher utility to the chosen bundle than to the rejected. Ordinal utility captures only ranking and not strength of preferences.

- **Cardinal utility** theories attach a significance to the magnitude of utility. The size of the utility difference between two bundles of goods is supposed to have some sort of significance.

Existence of a Utility Function

- Suppose preferences are complete, reflexive, transitive, continuous, and strongly monotonic.
- Then there exists a continuous utility function

$$u: \quad \Re_+^2 \to \Re$$

which represents those preferences.

The Utility Function

- A utility function is a function *u* assigning a real number to each consumption bundle so that for a pair of bundles *x* and *y*:

$$u(x) > u(y) \iff x \succ y\,,$$

$$u(x) = u(y) \iff x \sim y\,,$$

$$u(x) \geq u(y) \iff x \succsim y\,.$$

Examples of Utility Functions

Goods	The utility function
Normal	$u(x_1, x_2) = x_1^c x_2^d$, $c, d > 0$ - the Cobb-Douglas utility function
Perfect substitutes	$u(x_1, x_2) = a_1 x_1 + a_2 x_2$, $a_1, a_2 > 0$
Perfect complements	$u(x_1, x_2) = \min\{a_1 x_1, a_2 x_2\}$, $a_1, a_2 > 0$

The Quasilinear Utility Function

- The quasilinear (partly linear) utility function is linear in one argument.
- For example the utility function linear in good 2 is the following:

$$u(x_1, x_2) = v(x_1) + x_2$$

The Quasilinear Utility Function

- Specific examples of quasilinear utility would be:

$$u(x_1, x_2) = \sqrt{x_1} + x_2$$

or

$$u(x_1, x_2) = \ln x_1 + x_2$$

The Homothetic Utility Function

A function is called **homothetic** if it is a positive monotonic transformation of a homogenous function.

A function $u: \quad \Re^n \to \Re$ is **homogenous** of degree 1 if $u(\alpha x) = \alpha u(x)$ for all $\alpha > 0$.

A proportional increase in consumption of all goods yields a proportional increase in utility.

Example: $u(x_1, x_2) = \left(x_1^\rho + x_2^\rho\right)^{\frac{1}{\rho}}$ or $u(x_1, x_2) = x_1^c x_2^{1-c}$

The Homothetic Utility Function

A function $u(x)$ is homothetic if $u(x) = g(h(x))$ where g is a strictly increasing function and h is a homogenous of degree 1 function.

So a function is homothetic in x if it can be decomposed into inner function that is homogenous of degree one in x and an outer function monotonically increasing in its argument.

The Homothetic Utility Function

- Slopes of indifference curves are constant along a ray through the origin.

- Assuming that preferences can be represented by a homothetic function is equivalent to assuming that they can be represented by a function that is homogenous of degree 1 because a utility function is unique up to a positive monotonic transformation.

The Marginal Utility

Consider a consumer who is consuming some bundle of goods (x_1, x_2).
How does the consumer's utility change as the consumer is given a little more of one of goods.

The marginal utility with respect to good i is a change in utility due to an incremental change in consumption of good i holding consumption of other good fixed: $MU_i = \dfrac{\partial u(x)}{\partial x_i} > 0$, $i = 1, 2$.

The magnitude of marginal utility depends on the magnitude of utility.

The Marginal Rate of Substitution

- Suppose that we increase the amount of good *i*; how does the consumer have to change their consumption of good *j* in order to keep utility constant?

Let dx_i and dx_j be the changes in x_i and x_j ($i, j = 1, 2$, $i \neq j$).

$$\frac{dx_j}{dx_i} = -\frac{\partial u(x)}{\partial x_i} \bigg/ \frac{\partial u(x)}{\partial x_j}$$ - the marginal rate of substitution between goods i and j.

The Marginal Rate of Substitution

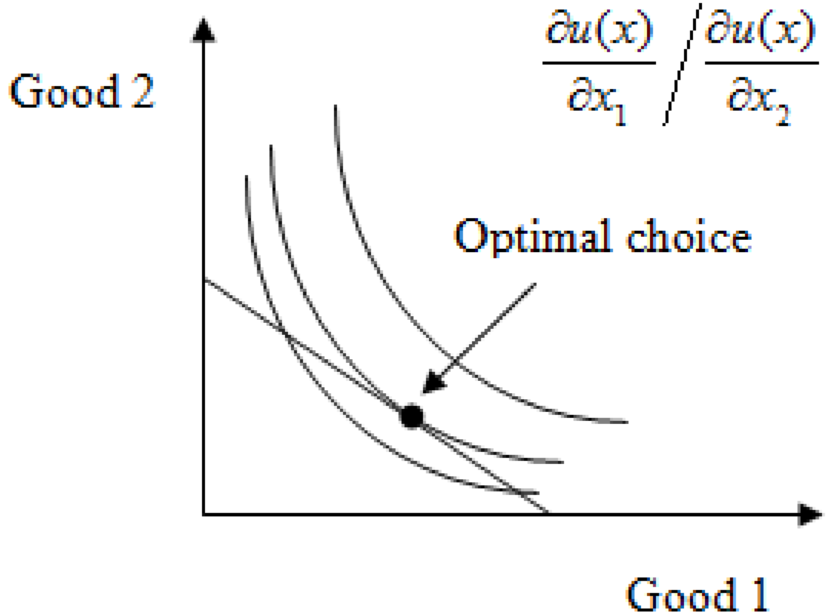
The marginal rate of substitution does not depend on the utility function chosen to represent the underlying preferences.

Let $v(u)$ be a monotonic transformation of utility.
The marginal rate of substitution for this utility function is

$$\frac{dx_j}{dx_i} = -\left(\frac{dv}{du}\frac{\partial u(x)}{\partial x_i}\right)\Bigg/\left(\frac{dv}{du}\frac{\partial u(x)}{\partial x_j}\right) = -\frac{\partial u(x)}{\partial x_i}\Big/\frac{\partial u(x)}{\partial x_j}.$$

The Optimal Choice

- Consumers choose the most preferred bundle from their budget sets.
- The optimal choice of consumer is that bundle in the consumer's budget set that lies on the highest indifference curve.



$$\frac{\partial u(x)}{\partial x_1} \Big/ \frac{\partial u(x)}{\partial x_2} = \frac{p_1}{p_2}$$

The Optimal Choice

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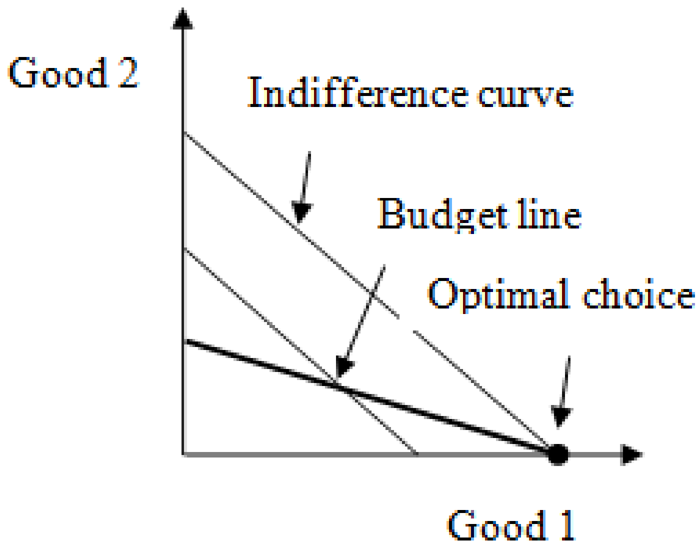
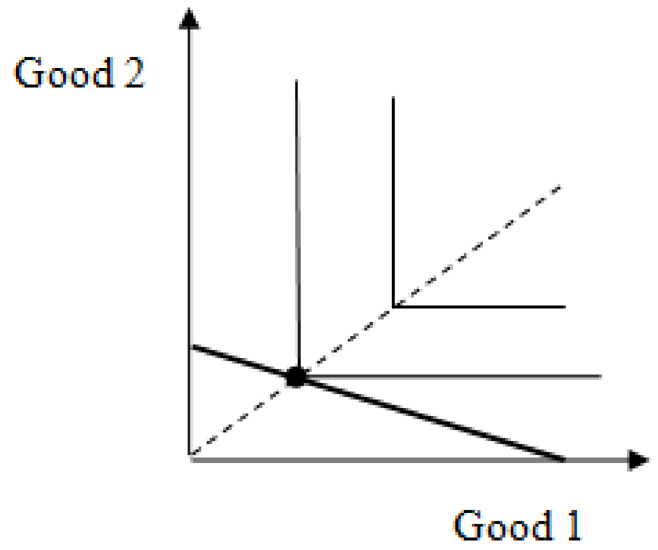
<tr><td>
</td><td>
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The Optimal Choice

$$u(x_1, x_2) = \left(\frac{1}{2}x_1 + 2\right)(x_2 + 4)$$

$$2x_1 + x_2 = 8$$

$$\begin{cases} \dfrac{x_2 + 4}{x_1 + 4} = \dfrac{2}{1} \\ 2x_1 + x_2 = 8 \end{cases}$$

$(1, 6)$

The Optimal Choice

- Utility functions

a) $u(x_1, x_2) = x_1 + x_2$,

b) $u(x_1, x_2) = 4x_1 + x_2$,

- Budget line

$$2x_1 + x_2 = 8$$

The Optimal Choice

$$u(x_1, x_2) = \min\{2x_1 + x_2, x_1 + 2x_2\}$$

$$2x_1 + 3x_2 = 10$$

$$\begin{cases} x_1 = x_2 \\ 2x_1 + 3x_2 = 10 \end{cases}$$

$$(2, 2)$$

The Utility Maximization

- The problem of utility maximization can be written as:

$$\max_{x_1, x_2} u(x_1, x_2)$$

$$\text{such that}$$

$$p_1 x_1 + p_2 x_2 = I$$

- Consumers seek to maximize utility subject to their budget constraint.

- The consumption levels which solve the utility maximization problem are the Marshallian demand functions.

The Lagrange Method

- The method starts by defining an auxiliary function known as the Lagrangean:

$$L(x_1, x_2, \lambda) = u(x_1, x_2) + \lambda\left(I - p_1 x_1 - p_2 x_2\right)$$

- The new variable λ is called a Lagrange multiplier since it is multiplied by constraint.

The Lagrange Method

The Lagrange's theorem says that an optimal choice $(\tilde{x}_1, \tilde{x}_2)$ must satisfy the three first-order conditions:

$$\frac{\partial L}{\partial x_1} = \frac{\partial u(\tilde{x}_1, \tilde{x}_2)}{\partial x_1} - \lambda p_1 = 0,$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial u(\tilde{x}_1, \tilde{x}_2)}{\partial x_2} - \lambda p_2 = 0,$$

$$\frac{\partial L}{\partial \lambda} = \left(I - p_1\tilde{x}_1 - p_2\tilde{x}_2\right) = 0.$$