Mathematical Economics

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Lecture 3

- Properties of the Demand Function: the Marginal Demand, the Price, Income and Cross Price Elasticity of Demand,

- Classification of Goods: Normal Goods, Inferior Goods, Ordinary Goods, Giffen Goods, Perfect Substitutes, Perfect Complements,
- The Total Change in Demand: The Substitution Effect and the Income Effect,
- Comparative Statics: Income Offer Curve, Price Offer Curves and Engel Curves
- The Indirect Utility Function: Definition and Properties,
- The Roy's Identity,

The Demand Function

- The value of $\tilde{x}$ that solves the utility maximization problem

$$\max_{x_1, x_2} u(x_1, x_2)$$

such that

$$p_1 x_1 + p_2 x_2 = I$$

is the **consumer's demanded bundle**.

- It expresses how much of each good the consumer desires at a given level of prices and income.

The Demand Function

- The function that relates p and I to the demanded bundle is called consumer's demand function:

$$\varphi: \quad \Re_+^3 \to \Re_+^2$$

$$\varphi: \quad (p, I) \to \varphi(p, I) = \tilde{x}$$

$$\varphi = (\varphi_1(p_1, p_2, I), \varphi_2(p_1, p_2, I))$$

Elasticities of Demand

- **Elasticity** is the ratio of the per cent change in one variable to the per cent change in other variable (is a measure of sensitivity of one variable to another).

Price elasticity of demand

Cross price elasticity of demand

Income elasticity of demand

Marginal Demands	**Elasticities of Demand**
$\frac{\partial \varphi_i(p,I)}{\partial p_i}$ - marginal demand of price	$\frac{\partial \varphi_i(p,I)}{\partial p_i} \times \frac{p_i}{\varphi_i}$ - price elasticity of demand
$\frac{\partial \varphi_i(p,I)}{\partial p_j}$ - cross marginal demand of price	$\frac{\partial \varphi_i(p,I)}{\partial p_j} \times \frac{p_j}{\varphi_i}$ - cross price elasticity of demand
$\frac{\partial \varphi_i(p,I)}{\partial I}$ - marginal demand of income	$\frac{\partial \varphi_i(p,I)}{\partial I} \times \frac{I}{\varphi_i}$ - income elasticity of demand

Note: $\frac{\partial \varphi_i(p,I)}{\partial p_i} \times \frac{p_i}{\varphi_i} \equiv \frac{d \ln(\varphi_i(p,I))}{d \ln(p_i)}$, $i = 1, 2$.

Elasticities of Demand

- **Price elasticity of demand** is defined as the measure of responsiveness in the quantity demanded for a good as a result of change in price of the same good. It is a measure of how consumers react to a change in price of a given good (a measure of the sensitivity of quantity demanded to changes in price).

- **Cross price elasticity of demand** measures the responsiveness of the quantity demanded of a good to a change in the price of another good.

- **Income elasticity of demand** measures the responsiveness of the quantity demanded of a good to the change in the income of the consumer.

The Classification of Goods – An Ordinary Versus a Giffen Goods

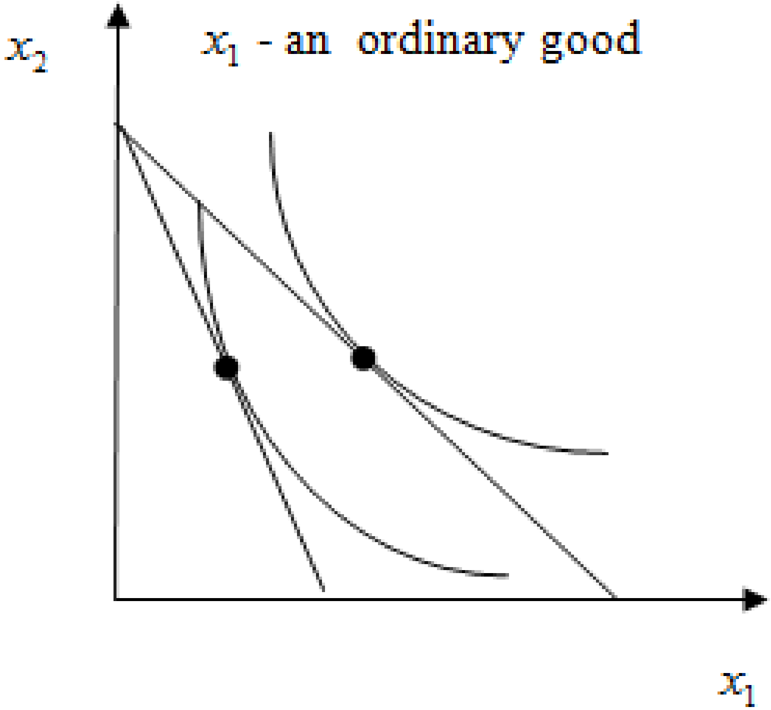
An ordinary good is one for which the demand decreases when its price increases

Good i is ordinary if $\frac{\partial \varphi_i(p,I)}{\partial p_i} < 0$ or $\frac{\partial \varphi_i(p,I)}{\partial p_i} \times \frac{p_i}{\varphi_i} < 0$.

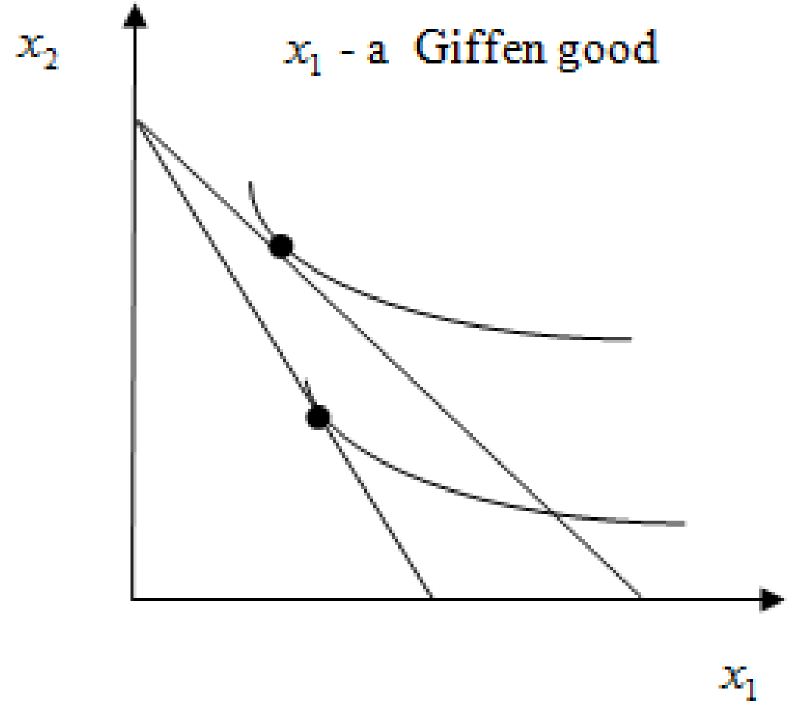
A Giffen good is one for which the demand increases when its price increases.

Good i is a Giffen good if $\frac{\partial \varphi_i(p,I)}{\partial p_i} > 0$ or $\frac{\partial \varphi_i(p,I)}{\partial p_i} \times \frac{p_i}{\varphi_i} > 0$.

The Classification of Goods – An Ordinary Versus a Giffen Goods



$$\frac{\partial \varphi_1(p, I)}{\partial p_1} \times \frac{p_1}{\varphi_1} < 0$$



$$\frac{\partial \varphi_1(p, I)}{\partial p_1} \times \frac{p_1}{\varphi_1} > 0$$

The Classification of Goods – Perfect Substitutes Versus Perfect Complements

If the demand for good i increases when the price of good j increases then good i is a **substitute** for good j :

$$\frac{\partial \varphi_i(p,I)}{\partial p_j} > 0 \text{ or } \frac{\partial \varphi_i(p,I)}{\partial p_j} \times \frac{p_j}{\varphi_i} > 0 .$$

If the two goods are substitutes then the cross elasticity of demand is positive. As the price of one goes up the quantity demanded of the other will increase.

If the demand for good i decreases when the price of good j increases then good i is a complement for good j:

$$\frac{\partial \varphi_i(p,I)}{\partial p_j} < 0 \text{ or } \frac{\partial \varphi_i(p,I)}{\partial p_j} \times \frac{p_j}{\varphi_i} < 0$$

(the cross elasticity of demand is negative)

The Classification of Goods – A Normal Versus an Inferior Goods

- **Normal goods**: if a good is normal, then the demand for it increases when income increases, and decreases when income decreases. The quantity demanded always changes in the same way as income changes.

$$\text{Good } i \text{ is normal if } \frac{\partial \varphi_i(p,I)}{\partial I} > 0 \text{ or } \frac{\partial \varphi_i(p,I)}{\partial I} \times \frac{I}{\varphi_i} > 0$$

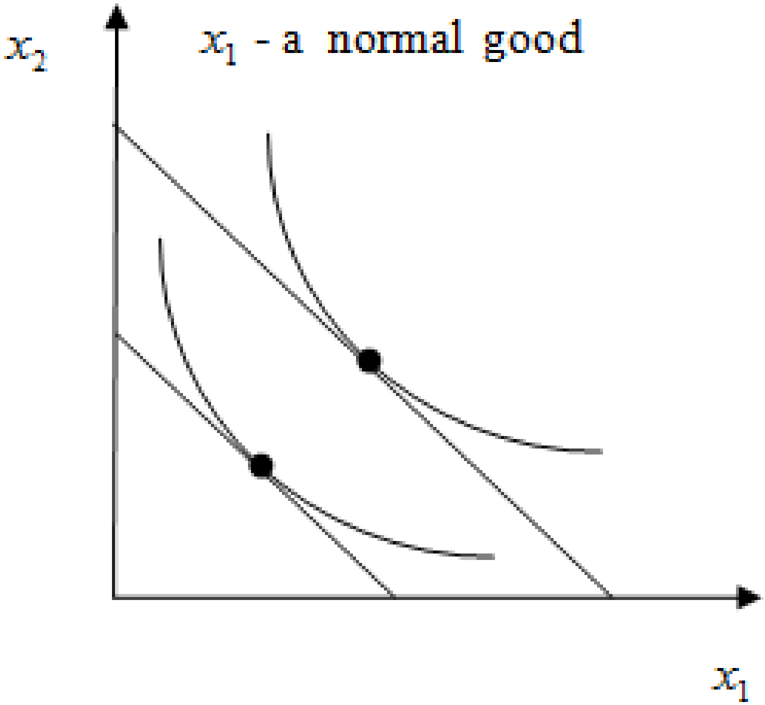
- If income elasticity of demand of a commodity is less than 1, it is **a necessity good.** If the elasticity of demand is greater than 1, it is a luxury or a superior good.

The Classification of Goods – A normal Versus an Inferior Goods

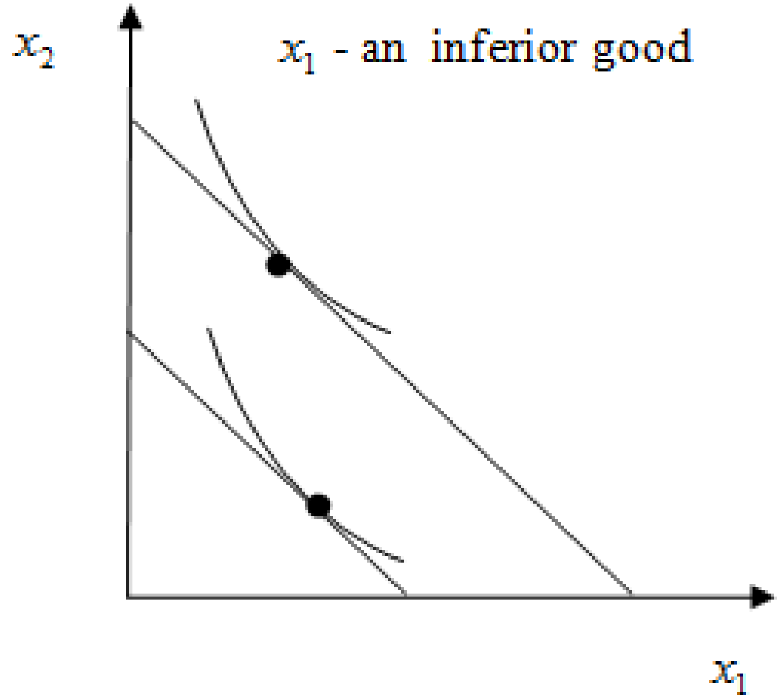
- **Inferior goods**: An inferior good is one for which the demand decreases when income increases.

$$\text{Good } i \text{ is inferior if } \frac{\partial \varphi_i(p,I)}{\partial I} < 0 \text{ or } \frac{\partial \varphi_i(p,I)}{\partial I} \times \frac{I}{\varphi_i} < 0.$$

The Classification of Goods – A normal Versus an Inferior Goods



$$\frac{\partial \varphi_1(p,I)}{\partial I} \times \frac{I}{\varphi_1} > 0$$



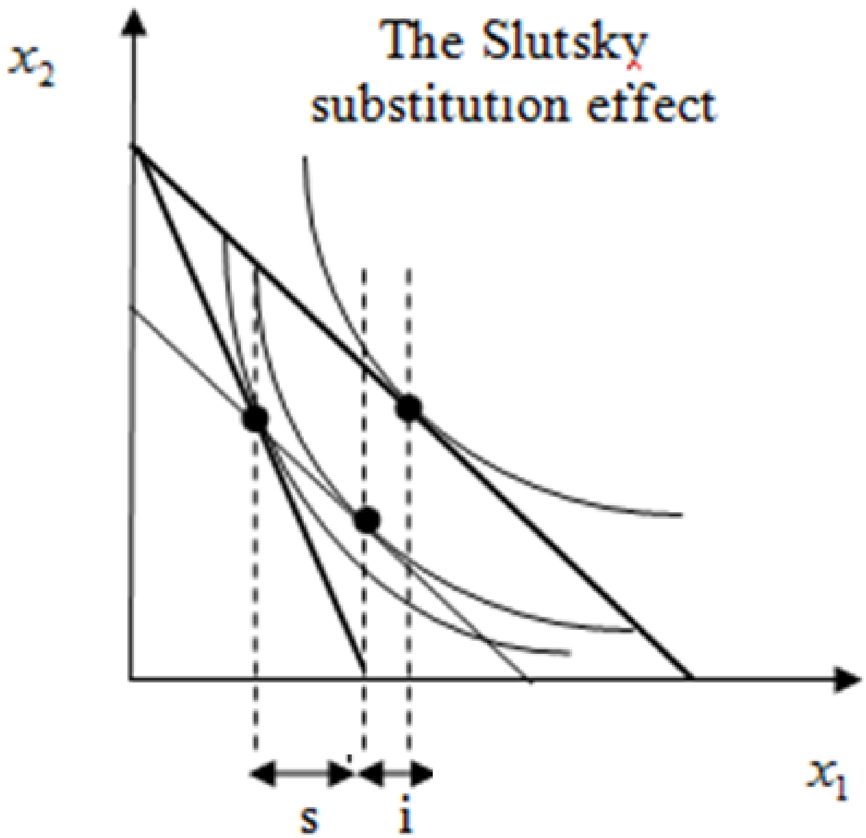
$$\frac{\partial \varphi_1(p,I)}{\partial I} \times \frac{I}{\varphi_1} < 0$$

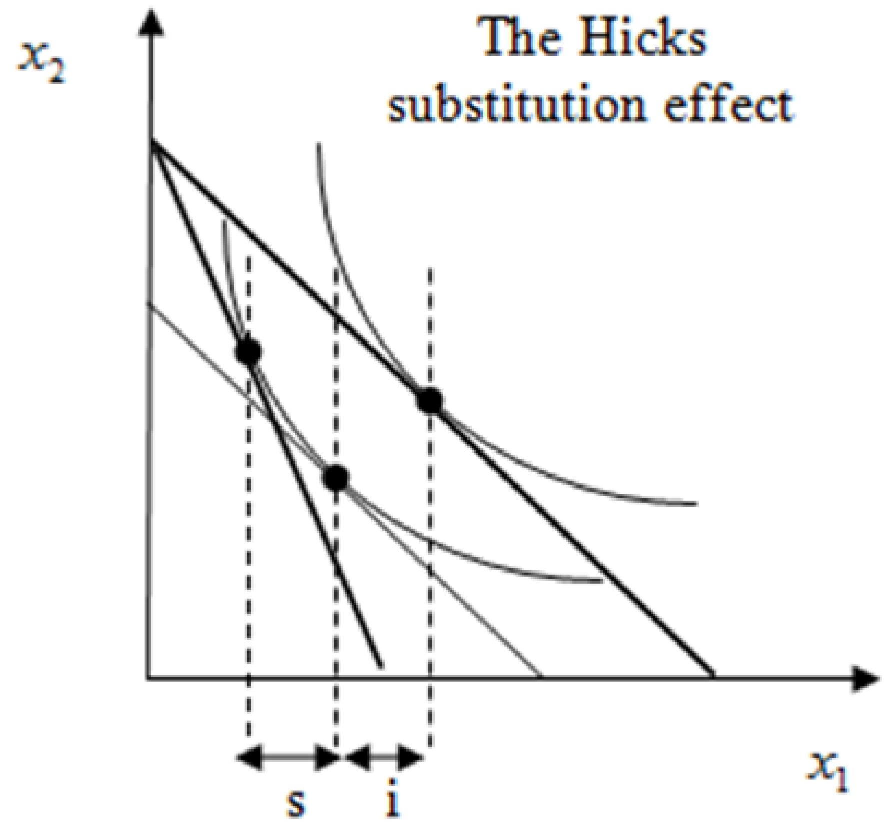
The Total Change in Demand: The Substitution Effect and the Income Effect

- When the price of good decreases, there will be two effects on consumption.
- The change in relative prices makes consumer want to consume more of the cheaper good. The increase in purchasing power due to the lower price may increase or decrease consumption, depending on whether the good is a normal good or an inferior good.

- The change in demand due to the change in relative prices is called **the substitution effect**; the change due to the change in purchasing power is called **the income effect**.

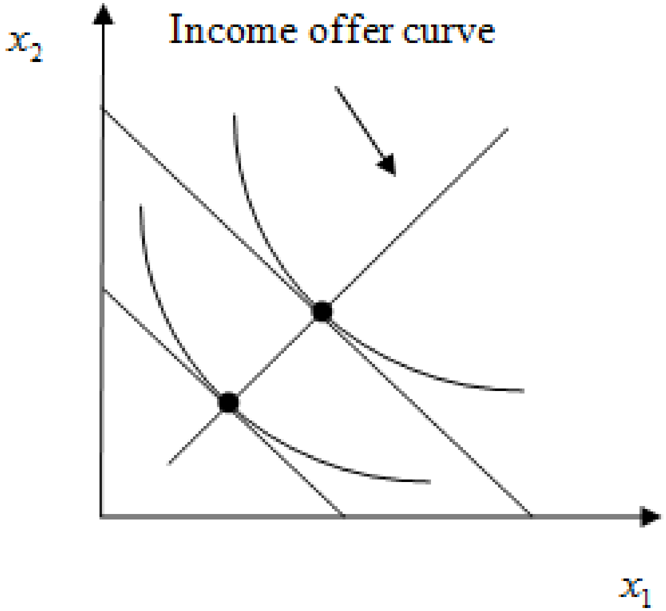
The Total Change in Demand: The Substitution Effect and the Income Effect

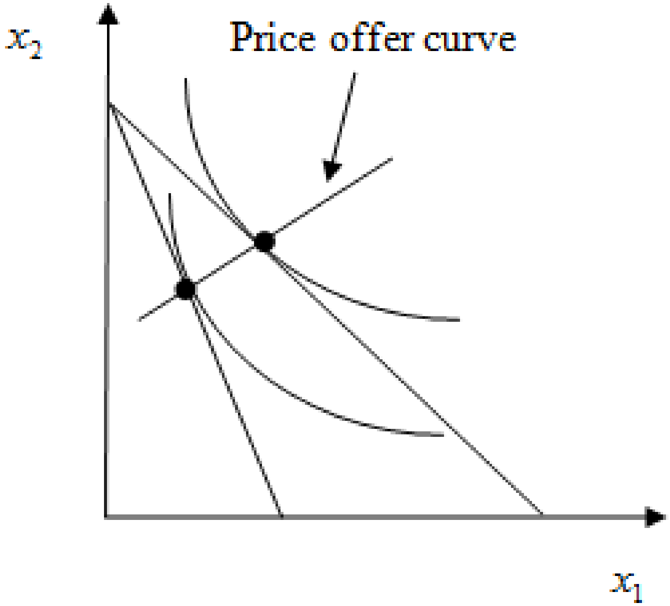




The Income Offer Curve (Income Expansion Path) and the Price Offer Curve

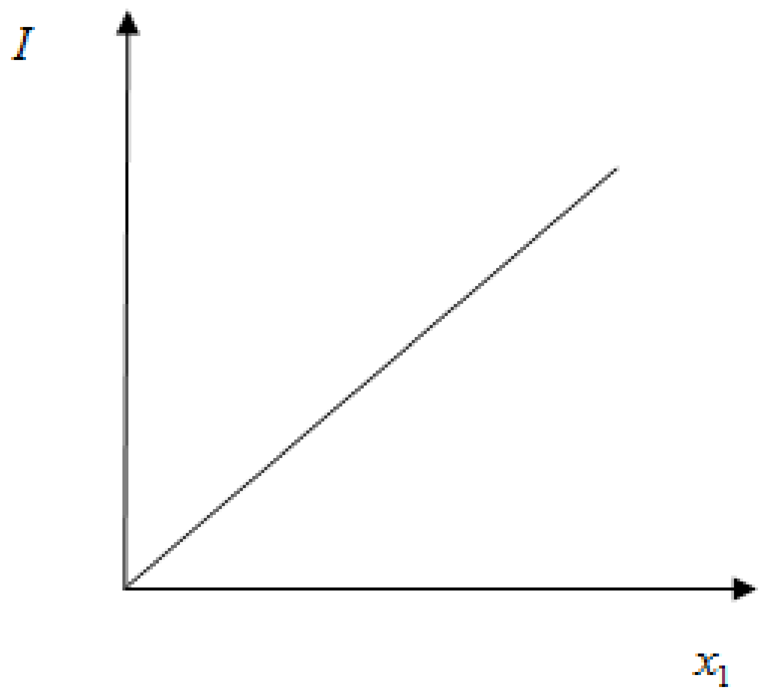
- The income offer curve depicts how consumption changes with income.
- The price offer curve represents the bundles that would be demanded at different prices for a given good.





The Engle Curve

- The Engle curve is a graph of the demand for a one of the goods as the function of income, with all prices being held constant.



The Indirect Utility Function

By substituting the consumption bundle x in $u(x)$ for the optimal bundle $\tilde{x}$ we obtain the indirect utility function $v = u(\tilde{x}) = u(\varphi(p, I))$.

The optimal consumption choice or the demand depends on prices and income i.e. $\tilde{x} = \varphi(p, I)$.
Thus the indirect utility function gives the maximum utility that can be achieved for certain prices and certain income.
The indirect utility is often denoted $v = v(p, I)$.

Example

In the Cobb-Douglas case utility function

$$u(x_1, x_2) = x_1^a x_2^{1-a}, \ a \in \langle 0,1 \rangle$$

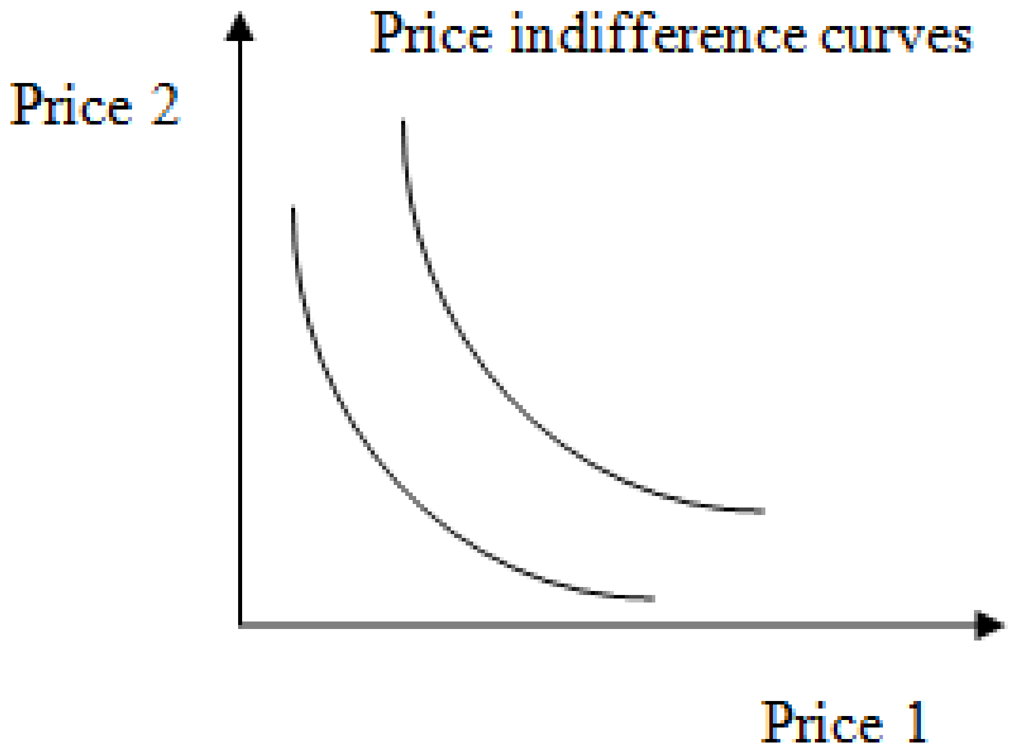
the indirect utility function is given by:

$$v(p, I) = \left(\frac{aI}{p_1}\right)^a \left(\frac{(1-a)I}{p_2}\right)^{1-a}$$

Properties of the Indirect Utility Function

1. $v(p, I)$ is nonincreasing in p (if $p' \geq p$, $v(p', I) \geq v(p, I)$) and nondecreasing in I;
2. $v(p, I)$ is homogenous of degree 0 in (p, I);
3. $v(p, I)$ is convex in p;
4. $v(p, I)$ is continuous in $p > 0$, $I > 0$.

Price indifference curve is all those prices such that $v(p, I) = k$ for some constant k.



Roy's Identity

If $\varphi(p, I)$ is the Marshallian demand function, then

$$\varphi_i(p, I) = -\frac{\partial v(p, I)/\partial p_i}{\partial v(p, I)/\partial I} \quad \text{for} \quad i = 1, 2$$

provided, of course, that the right-hand side is well defined and that $p_i > 0$ and $I > 0$.

- The Expenditure Minimization Problem,
- Properties of the Hicksian Demand Function,
- The Expenditure Function and its Properties,
- The Shephard's Lemma,
- Relationship between the Utility Maximization and the Expenditure Minimization Problem,
- The Slutsky Equation

The Expenditure Minimization Problem

- Instead of maximizing utility given a budget constraint we can consider the dual problem of minimizing the expenditure necessary to obtain a given utility level:

$$\min_{x_1, x_2} \sum_{i=1}^{2} p_i x_i$$

such that

$$u(x_1, x_2) = u$$

The Hicksian demand function

- The solution to this problem is the optimal consumption bundle as function of *p* and *u* i.e.

$$f\,(p, u)$$

- It is the expenditure-minimizing bundle necessary to achieve utility level *u* at prices *p*.

The Hicksian demand function

- The Hicksian demand function is sometimes called compensated demand function.
- This terminology comes from viewing the demand function as being constructed by varying prices and income so as to keep the consumer at fixed level of utility.
- Thus, the income changes are arranged to compensate for the price changes.

The Hicksian demand function

- The Hicksian demand functions are not directly observable since they depend on utility, which is not directly observable.
- The Marshallian demand functions expressed as function of prices and income are observable.

Example 1

Minimize expenditure $p_1 x_1 + p_2 x_2$

subject to $u(x_1, x_2) = x_1^a x_2^{1-a}$, $a \in \langle 0,1 \rangle$.

Solution:

$$f(p, u) = \left(\left(\frac{a}{1-a} \times \frac{p_2}{p_1} \right)^{1-a} \times u, \left(\frac{1-a}{a} \times \frac{p_1}{p_2} \right)^{a} \times u \right)$$

Properties of the Hicksian Demand Function

$$\frac{\partial f_i(p,u)}{\partial p_i} < 0, \qquad \forall p > 0$$

$$\frac{\partial f_i(p,u)}{\partial p_j} = \frac{\partial f_j(p,u)}{\partial p_i}$$

The Expenditure Function

The expenditure function $e(p, u)$ gives the minimum cost of achieving a fixed level of utility.
The minimal expenditure necessary

to reach u is $e(p, u) = \sum_{i=1}^{2} p_i f_i(p, u)$.

Example 2

$$e(p,u) = p_1 f_1 + p_2 f_2 = \frac{p_1^{a} p_2^{1-a} u}{a^{a}(1-a)^{1-a}}$$

Properties of the expenditure function

$e(p,u)$ is nondecreasing in p,

$e(p,u)$ is strictly increasing in u,

$e(p,u)$ is homogenous of degree 1 in p,

$e(p,u)$ in concave in p,

$e(p,u)$ is continuous in p, for $p > 0$,

$e\left(p, u(0)\right) = 0$,

$$\frac{\partial e(p,u)}{\partial p_i} = f_i(p,u) \text{ or } i = 1, 2 \text{ assuming}$$

the derivative exist and $p_i > 0$ – **the Shephard's lemma**.

Relationship between the Utility Maximization and the Expenditure Minimization Problem

The utility maximization problem	**The expenditure minimization problem**
$\max_{x_1, x_2} u(x_1, x_2)$ such that $p_1 x_1 + p_2 x_2 = I$	$\min_{x_1, x_2} p_1 x_1 + p_2 x_2$ such that $u(x_1, x_2) = u$
The Marshallian demand function $\varphi(p, I)$	**The Hicksian demand function** $f(p, u)$
The indirect utility $v(p, I)$	**The expenditure function** $e(p, u)$

$$u(x_1, x_2) = x_1^a x_2^{1-a},\ a \in (0,1)$$

$$\varphi(p, I) = \left(\frac{aI}{p_1}, \frac{(1-a)I}{p_2} \right) \qquad v(p, I) = \left(\frac{a}{p_1} \right)^a \left(\frac{(1-a)}{p_2} \right)^{1-a} I$$

$$f(p, u) = \left(\left(\frac{a}{1-a} \times \frac{p_2}{p_1} \right)^{1-a} \times u, \left(\frac{1-a}{a} \times \frac{p_1}{p_2} \right)^a \times u \right)$$

$$e(p, u) = \frac{p_1^a p_2^{1-a} u}{a^a (1-a)^{1-a}}$$

$e(p, v(p, I)) = I$ – the minimum expenditure to reach utility $u = v(p, I)$ is I.

$v(p, e(p, u)) = u$ – the maximum utility from income $e(p, u)$ is u.

$\varphi(p, I) = f(p, v(p, I))$ – the Marshallian demand at income I is the same as the Hicksian demand at utility $v = v(p, I)$.

$f(p, u) = \varphi(p, e(p, u))$ – the Hicksian demand at utility u is the same as the Marshallian demand at income $e(p, u)$.

The Slutsky Equation

$$f_i(p,u) = \varphi_i(p, e(p,u))$$

$$\frac{\partial f_i(p,u)}{\partial p_j} = \frac{\partial \varphi_i(p,e(p,u))}{\partial p_j} + \frac{\partial \varphi_i(p,e(p,u))}{\partial e(p,u)} \times \frac{\partial e(p,u)}{\partial p_j}$$

$$u = v(p,I), \qquad e(p,u) = e(p, v(p,I)) = I$$

$$\frac{\partial e(p,u)}{\partial p_j} = f_j(p,u) = f_j(p, v(p,I)) = \varphi_j(p,I)$$

$$\frac{\partial f_i(p,u)}{\partial p_j} = \frac{\partial \varphi_i(p,I)}{\partial p_j} + \frac{\partial \varphi_i(p,I)}{\partial I} \times \varphi_j(p,I)$$

$$\boxed{\frac{\partial \varphi_i(p,I)}{\partial p_j} = \underbrace{\frac{\partial f_i(p,u)}{\partial p_j}}_{\text{substitution effect}} - \underbrace{\frac{\partial \varphi_i(p,I)}{\partial I} \times \varphi_j(p,I)}_{\text{income effect}}}$$

The Hicks Decomposition of a Demand Change