Mathematical Economics

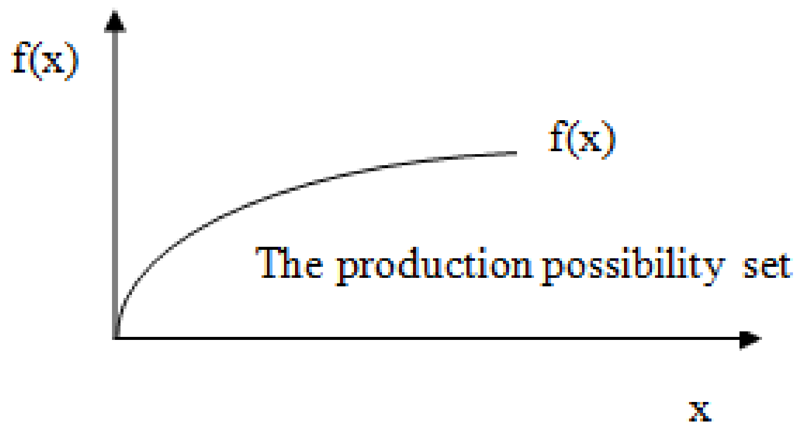
dr Wioletta Nowak

Lecture 4

- Production Function
- The Marginal Productivity of *i*-th Factor (the Marginal Product of Capital and the Marginal Product of Labour), Marginal Rate of Technical Substitution, Output Elasticities,
- Returns to Scale: Constant, Increasing, Decreasing,
- Elasticity of Substitution between Two Factors of Production,
- CES (Constant Elasticity of Substitution) Production Function,
- The Cobb-Douglas Production Function,

Production Function

- The production function $f(x_1, x_2)$ describes the maximum level of output that can be obtained for a given vector of inputs (x_1, x_2).

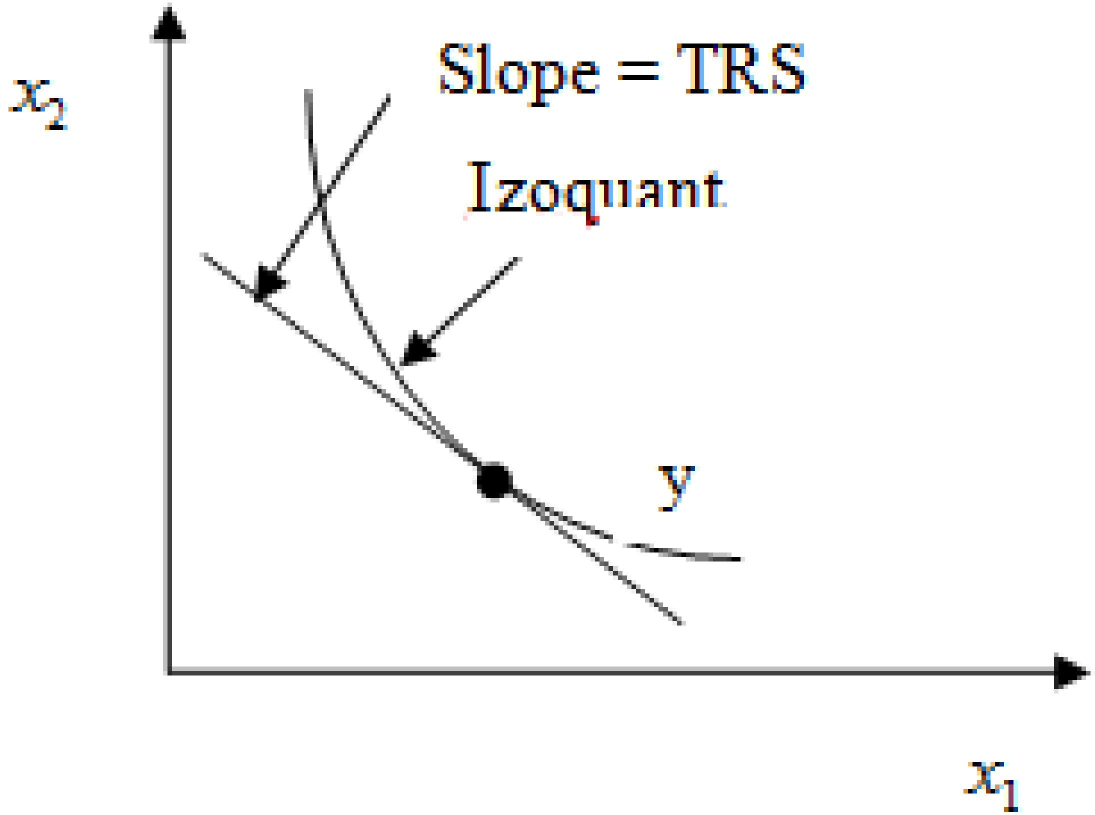


Note: x_1 - capital, x_2 - labour.

Isoquant

- Input combinations (x_1, x_2) that yield the same (maximum) level of output y define an isoquant.
- $x_2(x_1)$ is the function that tells us how much of x_2 it takes to produce y if we are using x_1 units of the other input.

Isoquant



The technical rate of substitution (TRS) between two inputs is a slope of the isoquant.

The technical rate of substitution

- TRS measures how one of the inputs must be adjust in order to keep output constant when another input changes.

TRS is the ratio of the marginal products of the inputs:

$$TRS = \frac{dx_2}{dx_1} = -\frac{\partial f / \partial x_1}{\partial f / \partial x_2}$$

$\frac{\partial f}{\partial x_1} > 0$ - **marginal product of capital (capital productivity)**.

It is an extra output produced by one more unit of capital.

$\frac{\partial f}{\partial x_2} > 0$ - **marginal product of labour (labour productivity)**.

Output Elasticities

If $y = f(x_1, x_2)$, the elasticity of y with respect to x_1 refers to the percentage change in y induced by a small percentage change in x_1.

$$\varepsilon_1 = \frac{\partial y}{\partial x_1}\frac{x_1}{y} \equiv \frac{\partial \ln y}{\partial \ln x_1}$$ - output elasticity of capital,

$$\varepsilon_2 = \frac{\partial y}{\partial x_2}\frac{x_2}{y}$$ - output elasticity of labour.

Elasticity of Substitution

Elasticity of substitution is a measure of the proportionate change in x_2/x_1 (labor to capital ratio) relative to the proportionate change in TRS along an isoquant.

$$\sigma = \frac{d(x_2/x_1)}{dTRS}\frac{TRS}{(x_2/x_1)} \equiv \frac{d\ln(x_2/x_1)}{d\ln(TRS)}$$

Returns to Scale

- **Constant returns to scale (CRS)**: Technology exhibits constant returns to scale if

$$f\left(t \cdot x_1, t \cdot x_2\right) = t \cdot f\left(x_1, x_2\right)$$

- If all inputs increase by a factor 2 then the new output is twice the previous output given.

Returns to Scale

- **Increasing returns to scale (IRS)**: Technology exhibits increasing returns to scale if

$$f\left(t \cdot x_1, t \cdot x_2\right) > t \cdot f\left(x_1, x_2\right)$$

- If all inputs increase by a factor 2 then the new output is more than twice the previous output given.

Returns to Scale

- **Decreasing returns to scale (DRS)**: Technology exhibits decreasing returns to scale if

$$f\left(t \cdot x_1, t \cdot x_2\right) < t \cdot f\left(x_1, x_2\right)$$

- If all inputs increase by a factor 2 then the new output is less than twice the previous output given.

Elasticity of Scale

- It may well happen that a technology exhibits increasing returns to scale for some values of x and decreasing returns to scale for other values.
- Thus a local measure of returns to scale is useful.
- The elasticity of scale measures the percentage increase in output due to a percentage increase in all inputs.

Elasticity of Scale

$$\varepsilon_t = \lim_{t \to 1} \frac{df(tx_1, tx_2)}{dt} \frac{t}{f(tx_1, tx_2)}$$

If $\varepsilon_t = 1$ production exhibits CRS.

If $\varepsilon_t > 1$ production exhibits IRS.

If $\varepsilon_t < 1$ production exhibits DRS.

The CES production function

- The constant elasticity of substitution (CES) production function has the form:

$$y(x_1, x_2) = A\left(ax_1^{\rho} + (1-a)x_2^{\rho}\right)^{\frac{1}{\rho}}$$

or

$$y(x_1, x_2) = A\left(ax_1^{-\rho} + (1-a)x_2^{-\rho}\right)^{-\frac{1}{\rho}},$$

$$0 \neq \rho > -1, \quad 0 < a < 1, \quad A > 0$$

The CES Function

- Marginal product of capital
- Marginal product of labour

$$\frac{\partial f}{\partial x_1} \qquad \frac{\partial f}{\partial x_2}$$

- Output elasticity of capital
- Output elasticity of labour

$$\varepsilon_1 = \frac{\partial y}{\partial x_1}\frac{x_1}{y} \qquad \varepsilon_2 = \frac{\partial y}{\partial x_2}\frac{x_2}{y}$$

- Elasticity of Substitution

$$\sigma = \frac{d\left(x_2/x_1\right)}{dTRS}\frac{TRS}{\left(x_2/x_1\right)} \equiv \frac{d\ln\left(x_2/x_1\right)}{d\ln(TRS)}$$

- Returns to Scale,
- Elasticity of Scale

$$\varepsilon_t = \lim_{t \to 1} \frac{df(tx_1, tx_2)}{dt}\frac{t}{f(tx_1, tx_2)}$$

The CES function contains several other production functions as special cases, depending on the value of the parameter ρ.

- The linear production function when $\rho = 1$.
- The Cobb-Douglas production function when $\rho = 0$ ($\lim_{\rho \to 0} A\left(ax_1^{\rho} + (1-a)x_2^{\rho}\right) = Ax_1^{a}x_2^{1-a}$)
- The Leontief- Koopmans production function when $\rho = -\infty$.

The CES Function

$$y = A\left(a \cdot x_1^{\rho} + (1-a)x_2^{\rho}\right)^{\frac{1}{\rho}}$$

$$y = A\left(a \cdot x_1^{-\rho} + (1-a)x_2^{-\rho}\right)^{-\frac{1}{\rho}}$$

$$A > 0, \quad a \in (0,1), \quad 0 \neq \rho > -1$$

$$y = A\left(a \cdot x_1^{\rho} + (1-a)x_2^{\rho}\right)^{\frac{1}{\rho}}$$

$$\frac{\partial y}{\partial x_1} = aA^{\rho}\left(\frac{y}{x_1}\right)^{1-\rho} \qquad \frac{\partial y}{\partial x_2} = (1-a)A^{\rho}\left(\frac{y}{x_2}\right)^{1-\rho}$$

$$\varepsilon_1 = aA^{\rho}\left(\frac{x_1}{y}\right)^{\rho} \qquad \varepsilon_2 = (1-a)A^{\rho}\left(\frac{x_2}{y}\right)^{\rho}$$

$$\varepsilon_1 + \varepsilon_2 = 1$$

$$y = A\left(a \cdot x_1^{\rho} + (1-a)x_2^{\rho}\right)^{\frac{1}{\rho}}$$

$$TRS = -\frac{a}{1-a}\left(\frac{x_2}{x_1}\right)^{1-\rho}$$

$$y = A\left(a \cdot x_1^{\rho} + (1-a)x_2^{\rho}\right)^{\frac{1}{\rho}}$$

$$\varepsilon_t = 1$$ **CRS**

$$\lim_{\rho \to 0} A\left(a \cdot x_1^{\rho} + (1-a)x_2^{\rho}\right)^{\frac{1}{\rho}} = A x_1^{a} x_2^{1-a}$$