

# Mathematical Economics

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## Lecture 5

- Neoclassical producer theory. Perfectly competitive firms.
- The profit function and profit maximization problem.
- Properties of the input demand and the output supply.
- Cost minimization problem. Definition and properties of the conditional factor demand and the cost function.
- Profit maximization with the cost function. Long and short run equilibrium.

## Perfect Competition

- Perfect competition describes a market in which there are many small firms, all producing homogeneous goods.
- No entry/exit barriers.
- The firm takes prices as a given in both its output and factor markets.
- Perfect information.
- Firms aim to maximize profits.

## Profit Maximization in the Long Run

$$\max_{x_1, x_2} \pi(x_1, x_2) = p \cdot f(x_1, x_2) - (v_1 x_1 + v_2 x_2)$$

or

$$\max_y \pi(y) = p \cdot y - c(y)$$

where  $c(y)$  is the cost function.

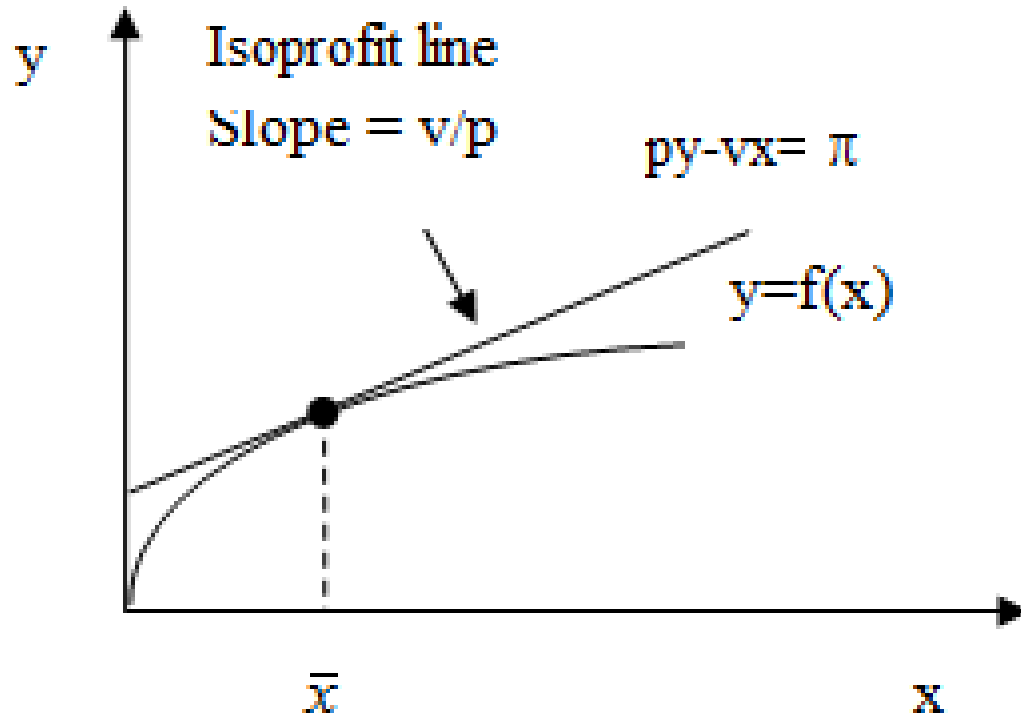
$$\max_{x_1, x_2} \pi(x_1, x_2) = p \cdot f(x_1, x_2) - (v_1 x_1 + v_2 x_2)$$

The first-order conditions for the profit maximization problem are

$$\begin{cases} \frac{\partial \pi(x_1, x_2)}{\partial x_1} = 0 \\ \frac{\partial \pi(x_1, x_2)}{\partial x_2} = 0 \end{cases} \Rightarrow \begin{cases} p \frac{\partial f(x_1, x_2)}{\partial x_1} = v_1 \\ p \frac{\partial f(x_1, x_2)}{\partial x_2} = v_2 \end{cases} \Rightarrow \bar{x} = (\bar{x}_1, \bar{x}_2)$$

The solution of firm's profit maximization problem  $\bar{x} = (\bar{x}_1, \bar{x}_2)$  is the factor demand function.

The first-order condition can be exhibited graphically



The tangency condition  $\frac{df(x)}{dx} = \frac{v}{p}$

## The Factor Demand Function $\bar{x}(p, v)$

- The factor demand function  $\bar{x}(p, v)$  gives the optimal choice of inputs as a function of the prices.
- Homogenous of degree 0 in  $(p, v)$ .
- Decreasing in factor prices.

## **The Supply Function $\bar{y}(p, v)$**

- The supply function of the firm  $\bar{y} = f(\bar{x}_1, \bar{x}_2)$ .
- Homogenous of degree 0 in  $(p, v)$ .
- Increasing in output price.



## The Profit Function $\pi(p, v)$

- The profit function  $\pi(\bar{x}_1, \bar{x}_2) = \pi(p, v)$
- Nondecreasing in output price  $p$ , nonincreasing in input prices  $(v_1, v_2)$ .
- Homogenous of degree 1 in  $(p, v)$ .
- Continuous in  $(p, v)$ .
- Convex in  $(p, v)$ .

# Cost Minimization in the Long Run

$$\min_{x_1, x_2} v_1 x_1 + v_2 x_2$$

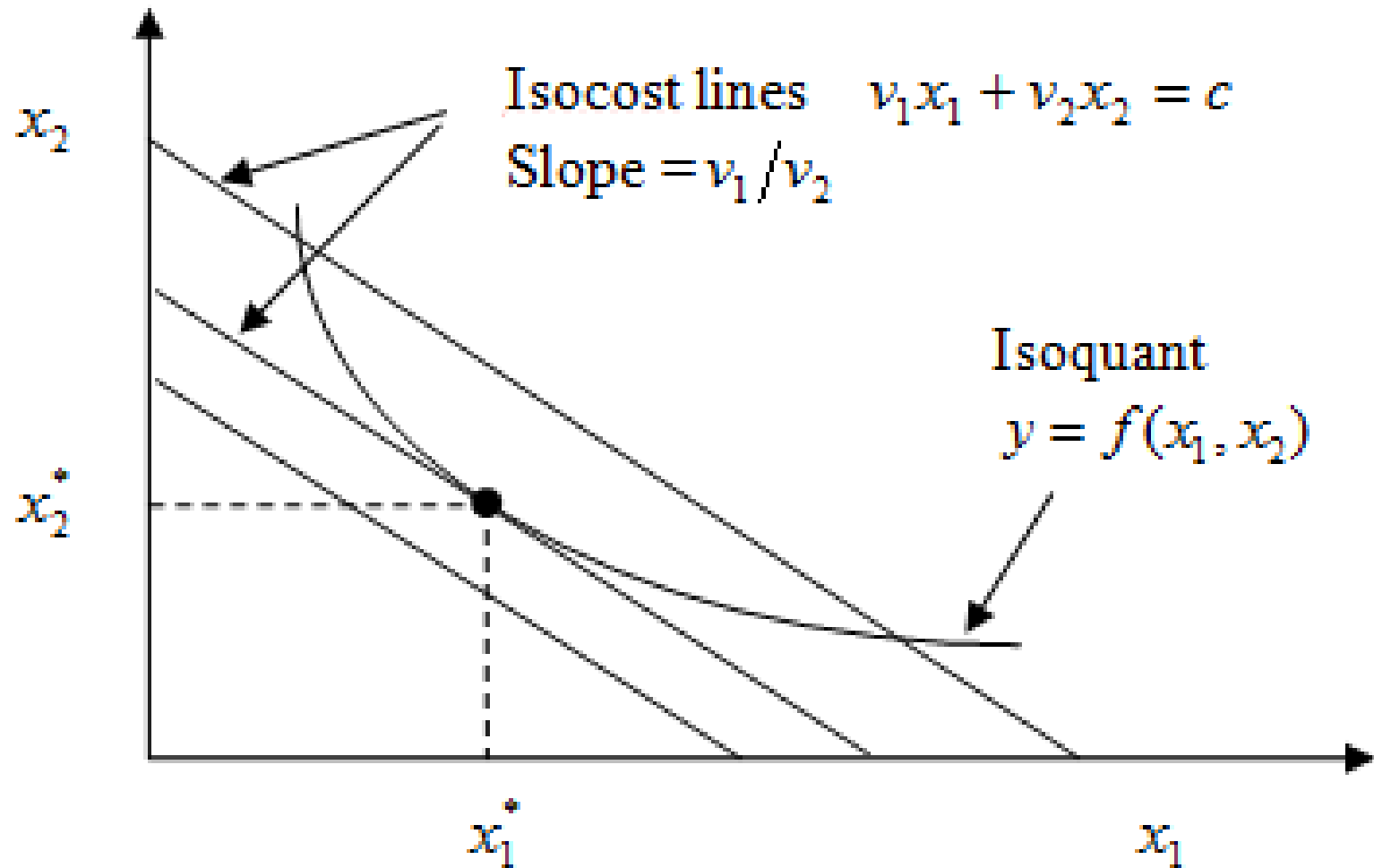
such that

$$y = f(x_1, x_2)$$

$$L(x_1, x_2, \lambda) = v_1 x_1 + v_2 x_2 + \lambda(y - f(x_1, x_2))$$

$$\left\{ \begin{array}{l} \frac{\partial L(x_1, x_2, \lambda)}{\partial x_1} = 0 \\ \frac{\partial L(x_1, x_2, \lambda)}{\partial x_2} = 0 \\ \frac{\partial L(x_1, x_2, \lambda)}{\partial \lambda} = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} v_1 = \lambda \frac{\partial f(x_1, x_2)}{\partial x_1} \\ v_2 = \lambda \frac{\partial f(x_1, x_2)}{\partial x_2} \\ y = f(x_1, x_2) \end{array} \right.$$

The cost minimization problem can be exhibited graphically



## **The Conditional Factor Demand Function $x^*(v, y)$**

The level of each input that solves the cost minimization problem are called conditional input demands and are denoted

$$x^*(v, y) = (x_1^*(v, y), x_2^*(v, y))$$

## The Cost Function $c(v, y)$

- Inserting  $x_1^*$  and  $x_2^*$  into the total cost expression yields the cost function  $c(v, y) = v_1 x_1^* + v_2 x_2^*$ .
- $c(v, y)$  measures the minimum costs of producing a given level of output at given factor prices.
- $c(v, y)$  is increasing in  $y$  and nondecreasing in factor prices.
- $c(v, y)$  is homogenous of degree 1 in  $v$ .
- $c(v, y)$  is concave in  $v$ .

# Profit Maximization with the Cost Function

$$\max_y \pi(y) = p \cdot y - c(y)$$

$$\frac{d\pi(y)}{dy} = 0 \quad \Rightarrow \quad p = \frac{dc(y)}{dy} \quad \Rightarrow \quad \bar{y}$$

## Example 1

Derive the factor demand function, the supply function, the profit function, the conditional factor demand function and the cost function of the firm with the Cobb-Douglas

production technology  $y = 4x_1^{\frac{1}{4}}x_2^{\frac{1}{2}}$ .

Solve profit maximization problem with the cost function.

$$\max_{x_1, x_2} \pi(x_1, x_2) = 4px_1^{\frac{1}{4}}x_2^{\frac{1}{2}} - (v_1x_1 + v_2x_2)$$

- $\bar{x}_1 = \frac{4p^4}{v_1^2 v_2^2}$ ,  $\bar{x}_2 = \frac{8p^4}{v_1 v_2^3}$  - the factor demand function,
- $\bar{y} = \frac{16p^3}{v_1 v_2^2}$  - the supply function,
- $\pi(p, v) = \frac{4p^4}{v_1 v_2^2}$  - the profit function,



$$\min_{x_1, x_2} v_1 x_1 + v_2 x_2$$

such that

$$y = 4x_1^{\frac{1}{4}} x_2^{\frac{1}{2}}$$

- $x_1^* = \left(\frac{v_2}{2v_1}\right)^{\frac{2}{3}} \left(\frac{y}{4}\right)^{\frac{4}{3}}, \quad x_2^* = \left(\frac{2v_1}{v_2}\right)^{\frac{1}{3}} \left(\frac{y}{4}\right)^{\frac{4}{3}}$

- the conditional factor demand function

- $c(v, y) = 3 \cdot 2^{-\frac{2}{3}} \cdot v_1^{\frac{1}{3}} \cdot v_2^{\frac{2}{3}} \cdot \left(\frac{y}{4}\right)^{\frac{4}{3}}$

- the cost function

$$\max_y \pi(y) = p \cdot y - 3 \cdot 2^{\frac{2}{3}} \cdot v_1^{\frac{1}{3}} \cdot v_2^{\frac{2}{3}} \cdot \left(\frac{y}{4}\right)^{\frac{4}{3}}$$

- $\bar{y} = \frac{16p^3}{v_1 v_2^2}$ .

The cost function for

- The Leontief technology

$$f(x_1, x_2) = \min \{a_1 x_1, a_2 x_2\} \text{ is given by } c(v, y) = \left( \frac{v_1}{a_1} + \frac{v_2}{a_2} \right) \cdot y$$

- The linear technology

$$f(x_1, x_2) = a_1 x_1 + a_2 x_2 \text{ is given by } c(v, y) = \min \left\{ \frac{v_1}{a_1}, \frac{v_2}{a_2} \right\} \cdot y$$

# Profit Maximization and Cost Minimization in the Short Run

Suppose that in the short run factor 1 is fixed at some predetermined level  $\hat{x}_1$ .  
 $y = f(\hat{x}_1, x_2)$  - the production function in short run.

$$1. \max_{x_2} \pi(\hat{x}_1, x_2) = p \cdot f(\hat{x}_1, x_2) - (v_1 \hat{x}_1 + v_2 x_2)$$

$$\frac{\partial \pi(\hat{x}_1, x_2)}{\partial x_2} = 0 \Rightarrow p \frac{\partial f(\hat{x}_1, x_2)}{\partial x_2} = v_2 \Rightarrow \bar{x}_2$$

$$2. \min_{x_2} v_1 \hat{x}_1 + v_2 x_2 \text{ such that } y = f(\hat{x}_1, x_2) \Rightarrow x_2^* = x_2^*(v, y, \hat{x}_1)$$

$$c(v, y) = v_1 \hat{x}_1^* + v_2 x_2^*$$

- Total Costs, Average Costs,
- Marginal Costs,
- Long-run Costs, Short-run Costs,
- Cost Curves, Long-run and Short-run Cost Curves,
- Monopoly

# Total Costs

total costs (TC) = variable costs (VC) + fixed costs (FC)

$$TC = VC + FC \quad (TC \equiv c(y))$$

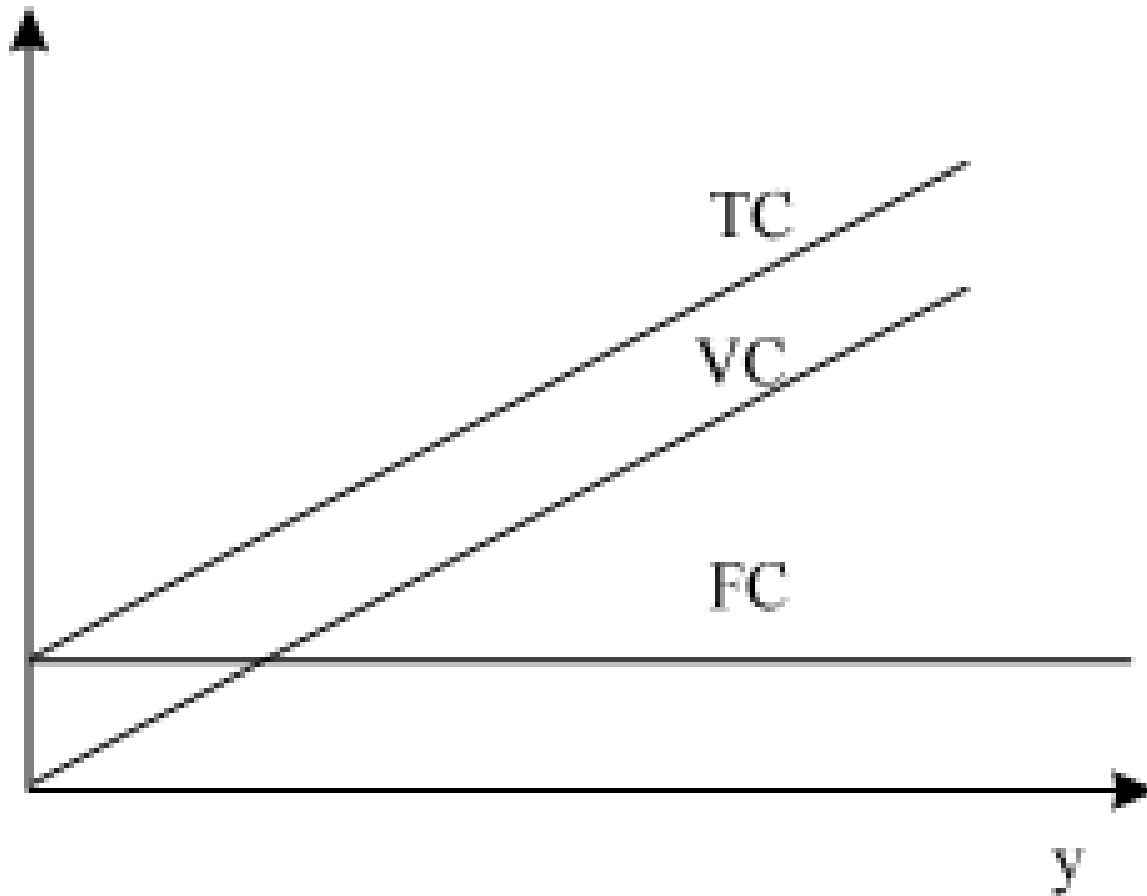
$$FC \equiv TC(0)$$

$$TC = \alpha y + \beta,$$

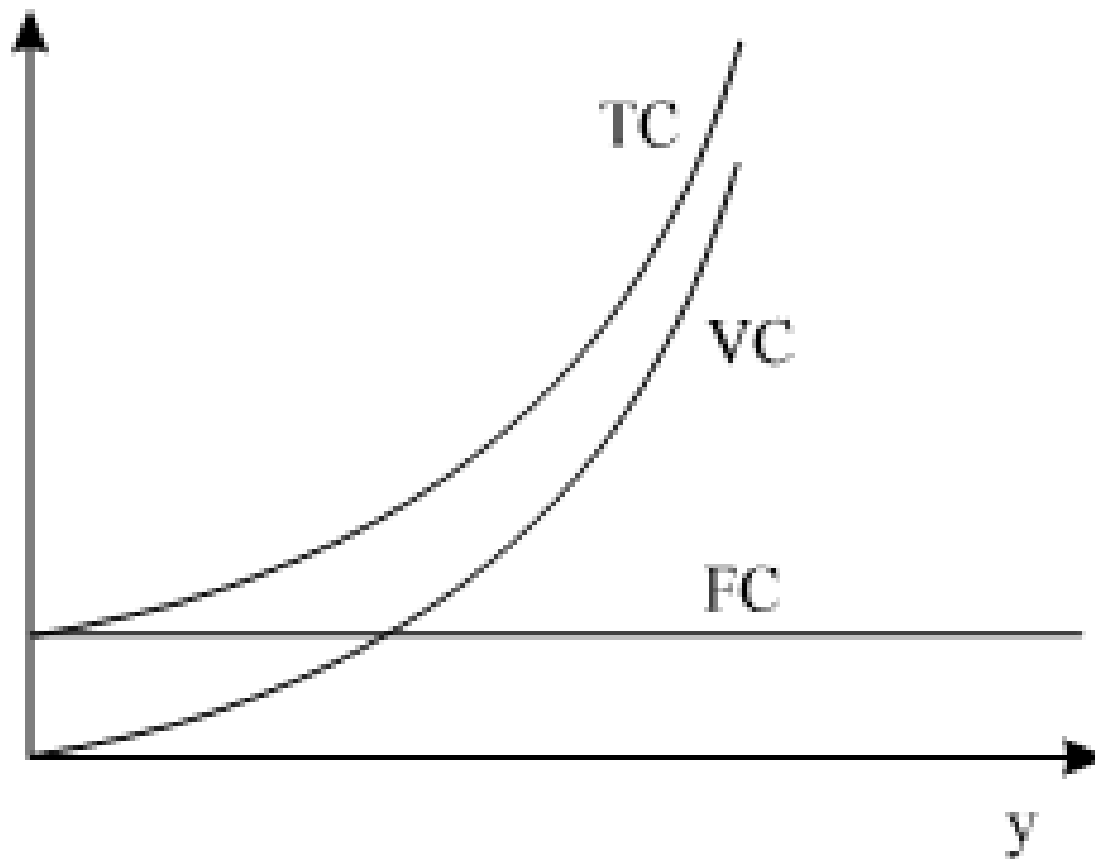
$$VC = \alpha y,$$

$$FC = \beta,$$

$$\alpha, \beta > 0$$

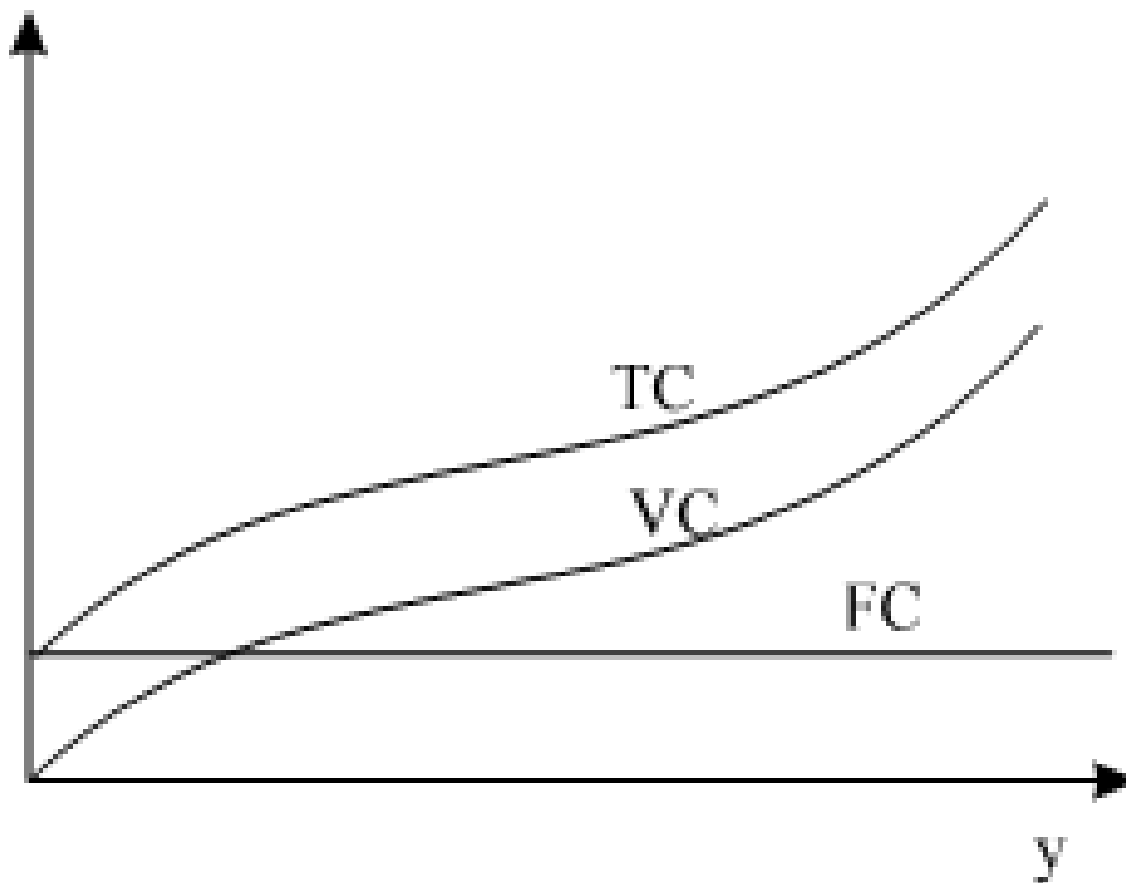


$$TC = \alpha y^2 + \beta y + \gamma, \quad VC = \alpha y^2 + \beta y, \quad FC = \gamma, \quad \alpha, \beta, \gamma > 0$$

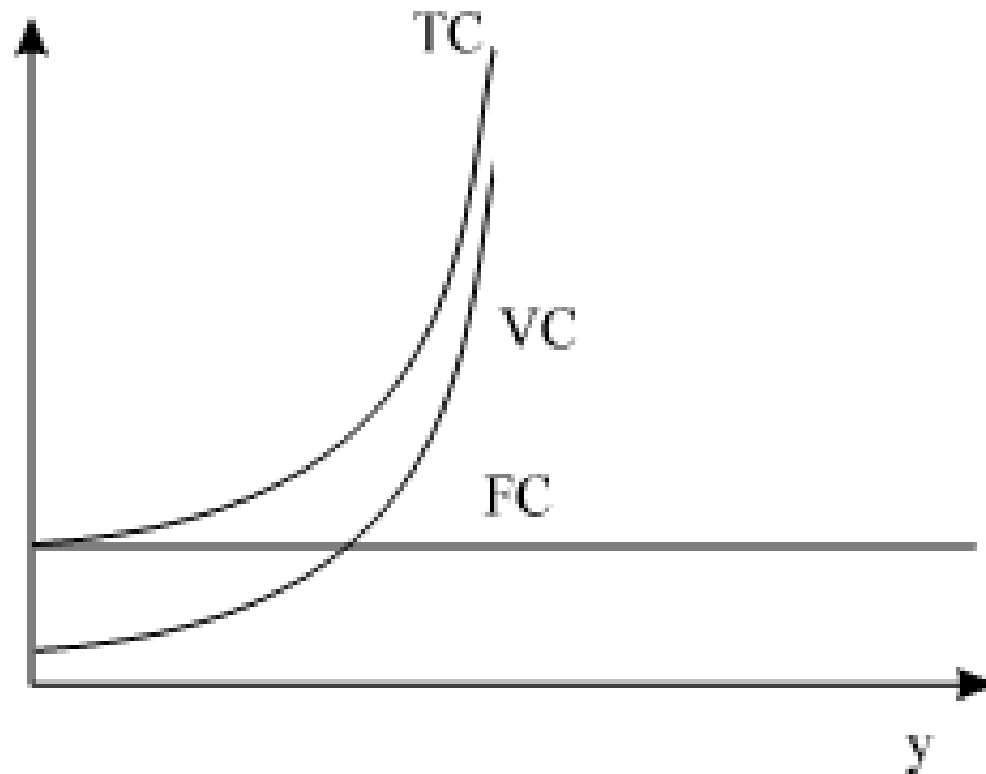




$$TC = \alpha y^3 + \beta y^2 + \gamma y + \delta, \quad VC = \alpha y^3 + \beta y^2 + \gamma y, \quad FC = \delta$$
$$\alpha, \gamma, \delta > 0, \quad \beta < 0, \quad \beta^2 < 3\alpha\gamma$$

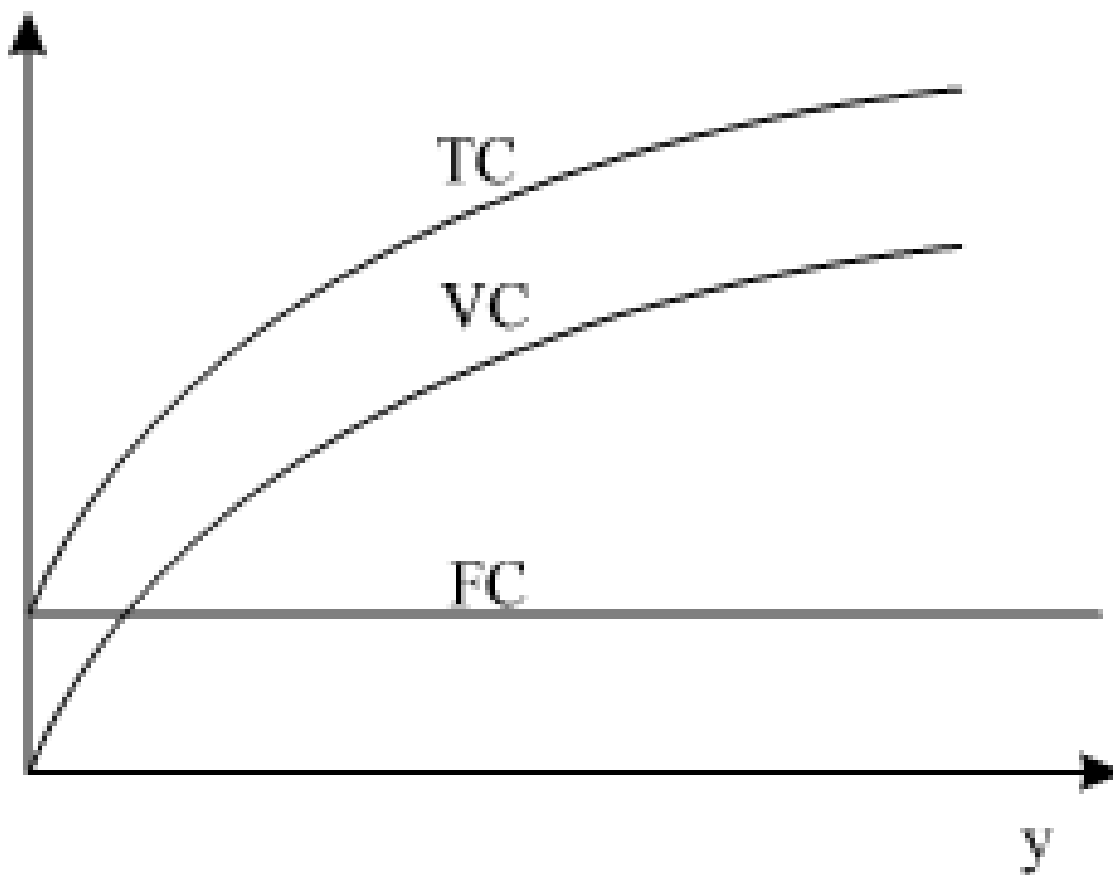


$$TC(y) = \alpha e^{\beta y}, \quad VC(y) = \alpha e^{\beta y}, \quad FC = \alpha, \quad \alpha, \beta > 0$$



Note:  $FC = TC(0) = \alpha e^{\beta \cdot 0} = \alpha e^{\beta \cdot 0} = \alpha \cdot 1$

$$TC(y) = \alpha y^\beta + \gamma, \quad VC(y) = \alpha y^\beta, \quad FC = \gamma, \quad \alpha, \gamma > 0, \quad 0 < \beta < 1$$



## Average Costs

- The average cost function measures the cost per unit of output

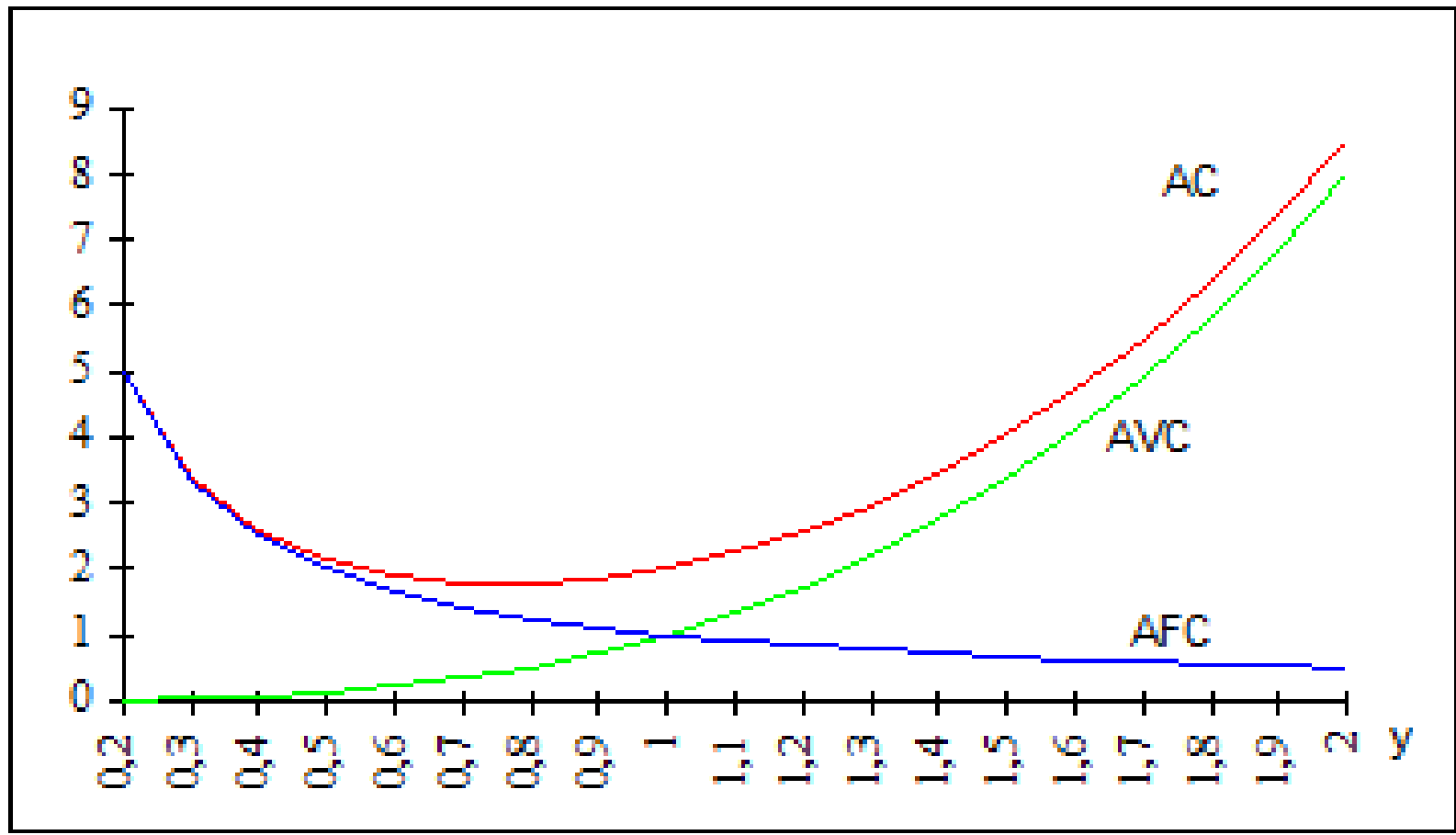
$$AC(y) = \frac{TC(y)}{y}$$

- Average costs (AC) = average variable costs (AVC) + average fixed costs (AFC)

$$AC(y) = \frac{VC(y)}{y} + \frac{FC(y)}{y} = AVC(y) + AFC(y)$$

- Average fixed costs always decline with output, while average variable costs tend to increase. The net result is a U-shaped average cost curve.

$$TC = y^4 + 1, \quad AC = y^3 + \frac{1}{y}, \quad AVC = y^3, \quad AFC = \frac{1}{y}$$



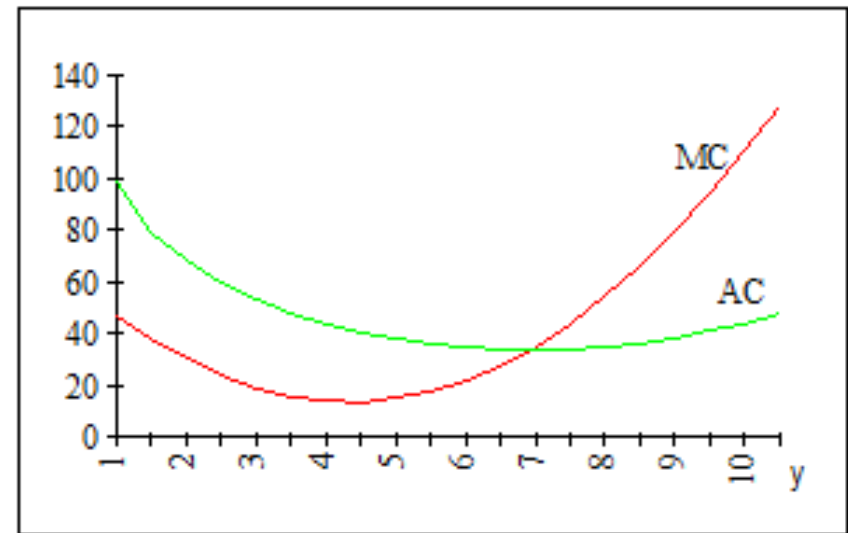
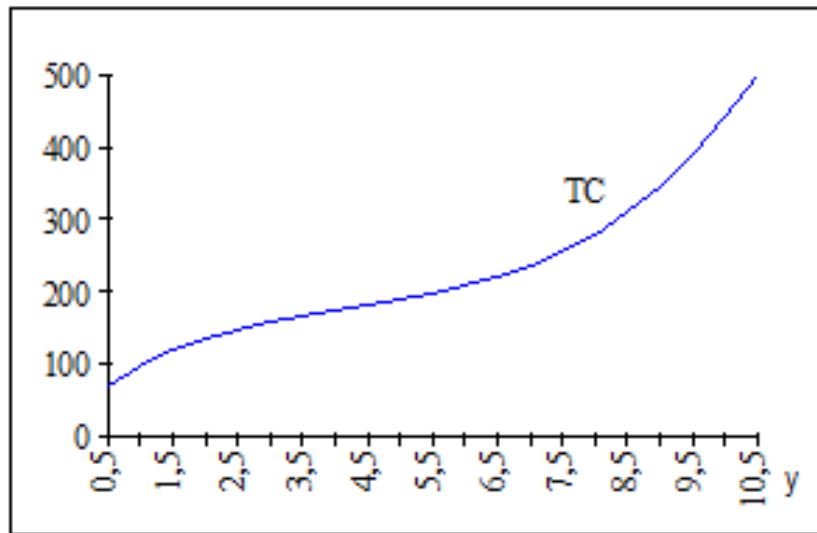
## Marginal Costs

$$MC = \frac{dTC(y)}{dy}$$

- The marginal cost curve lies below the average cost curve when average cost is decreasing, and above when they are increasing.
- The marginal costs are equal average costs at the point of minimum average costs.

$$TC(y) = y^3 - 13y^2 + 70y + 40, \quad AC(y) = y^2 - 13y + 70 + \frac{40}{y},$$

$$MC(y) = 3y^2 - 26y + 70$$



# Long-run Average Costs and Short-run Average Costs

$$\min_{x_1, x_2} (v_1 x_1 + v_2 x_2)$$

such that

$$y = Ax_1^c x_2^d, \quad A, c, d > 0$$

$$x_1^* = A^{-\frac{1}{c+d}} \left( \frac{cv_2}{dv_1} \right)^{\frac{d}{c+d}} y^{\frac{1}{c+d}}, \quad x_2^* = A^{-\frac{1}{c+d}} \left( \frac{cv_2}{dv_1} \right)^{-\frac{c}{c+d}} y^{\frac{1}{c+d}}$$

$$TC(y) \equiv c(y) = K_L y^{\frac{1}{c+d}}$$

$$K_L = A^{-\frac{1}{c+d}} \left\{ \left( \frac{c}{d} \right)^{\frac{d}{c+d}} + \left( \frac{d}{c} \right)^{\frac{c}{c+d}} \right\} v_1^{\frac{c}{c+d}} v_2^{\frac{d}{c+d}}$$



# Long-run Average Costs and Short-run Average Costs

$\hat{x}_2$  - the fixed level of input 2

$$c(y) = \min_{x_1} (v_1 x_1 + v_2 \hat{x}_2)$$

such that

$$y = Ax_1^c \hat{x}_2^d$$

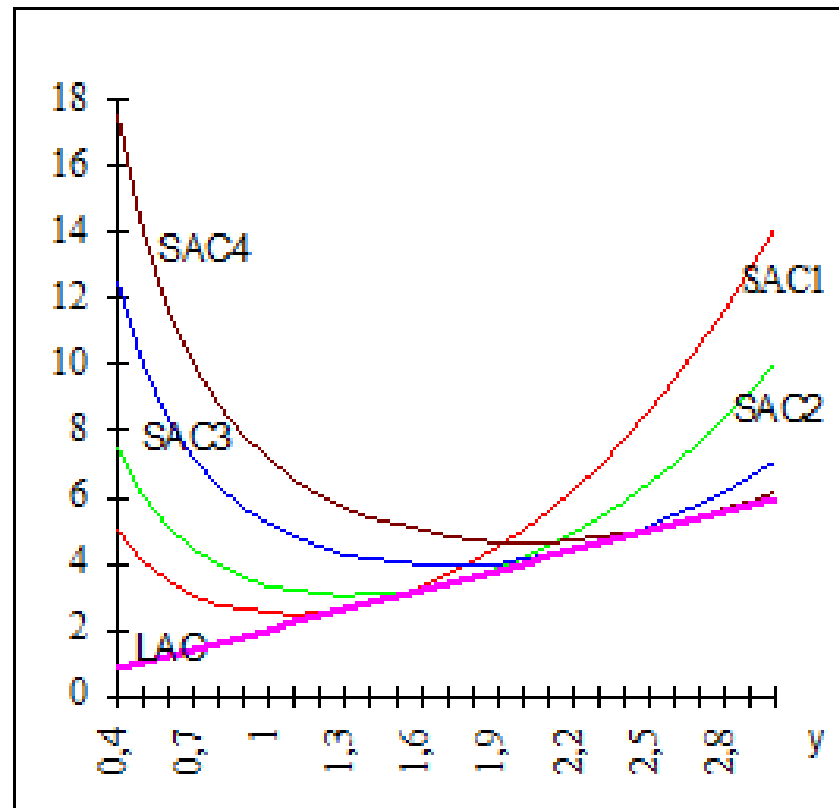
$$x_1^* = A^{-\frac{1}{c}} y^{\frac{1}{c}} \hat{x}_2^{-\frac{d}{c}}$$

$$c(y) = K_S y^{\frac{1}{c}} + F_S \quad K_S = A^{-\frac{1}{c}} v_1 \hat{x}_2^{-\frac{d}{c}}, \quad F_S = v_2 \hat{x}_2$$

$$LAC = K_L y^{\frac{1-c-d}{c+d}} \quad LMC = \frac{K_L}{c+d} y^{\frac{1-c-d}{c+d}}$$

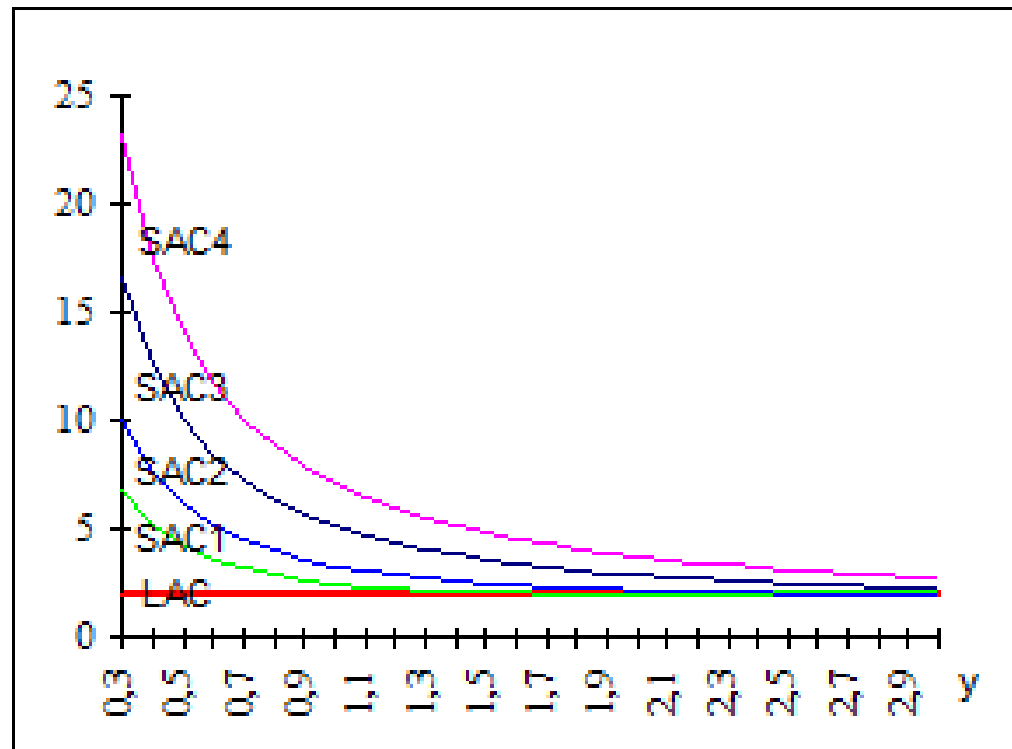
$$SAC = K_S y^{\frac{1-c}{c}} + \frac{F_S}{y} \quad SMC = \frac{K_S}{c} y^{\frac{1-c}{c}}$$

$$y = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}} \quad LAC = 2y, \quad SAC = \frac{y^3}{\hat{x}_2} + \frac{\hat{x}_2}{y}$$



$SAC1: \hat{x}_2 = 2, SAC2: \hat{x}_2 = 3, SAC3: \hat{x}_2 = 5, SAC4: \hat{x}_2 = 7$

$$y = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}, \quad LAC = 2, \quad SAC = \frac{y}{\hat{x}_2} + \frac{\hat{x}_2}{y}$$

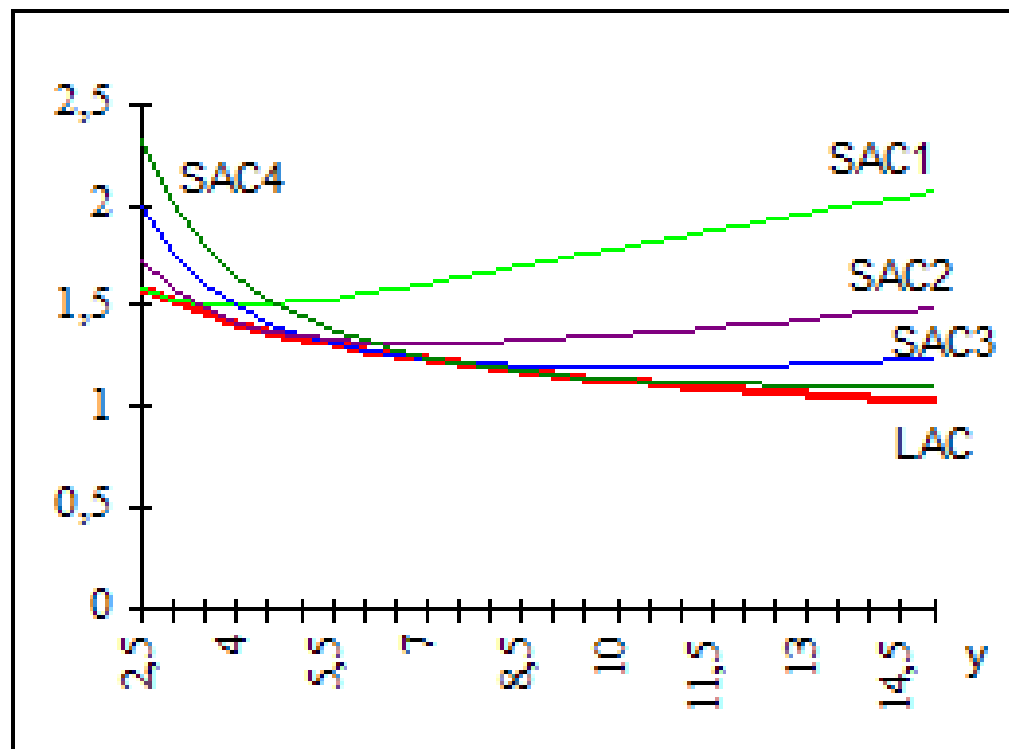


$$SAC1: \hat{x}_2 = 2, \quad SAC2: \hat{x}_2 = 3, \quad SAC3: \hat{x}_2 = 5, \quad SAC4: \hat{x}_2 = 7$$

$$y = x_1^2 x_2^2$$

$$LAC = 2y^{-\frac{1}{4}}$$

$$SAC = \frac{\sqrt{y}}{\hat{x}_2} + \frac{\hat{x}_2}{y}$$



SAC1:  $\hat{x}_2 = 2$ , SAC2:  $\hat{x}_2 = 3$ , SAC3:  $\hat{x}_2 = 4$ , SAC4:  $\hat{x}_2 = 5$

# Monopoly

- Monopoly is a price-maker.
- A monopolist has market power in the sense that amount of output that is able to sell responds continuously as a function of the price it charges.

The monopolist's profit maximization problem can be posed as

$$\max_{x_1, x_2} \pi(x_1, x_2) = r(x_1, x_2) - c(x_1, x_2)$$

where  $r(x_1, x_2) = p(f(x_1, x_2)) \cdot f(x_1, x_2)$  ;  
 $c(x_1, x_2) = v_1(x_1) \cdot x_1 + v_2(x_2) \cdot x_2$

The first-order conditions for the profit maximization problem are

$$\begin{cases} \frac{\partial \pi(x_1, x_2)}{\partial x_1} = 0 \\ \frac{\partial \pi(x_1, x_2)}{\partial x_2} = 0 \end{cases} \Rightarrow \begin{cases} \frac{\partial r(x_1, x_2)}{\partial x_1} = \frac{\partial c(x_1, x_2)}{\partial x_1} \\ \frac{\partial r(x_1, x_2)}{\partial x_2} = \frac{\partial c(x_1, x_2)}{\partial x_2} \end{cases}$$

$$\bar{x} = (\bar{x}_1, \bar{x}_2)$$



The monopolist's profit maximization problem can be posed as

$$\max_y \pi(y) = r(y) - c(y) = p(y) \cdot y - c(y)$$

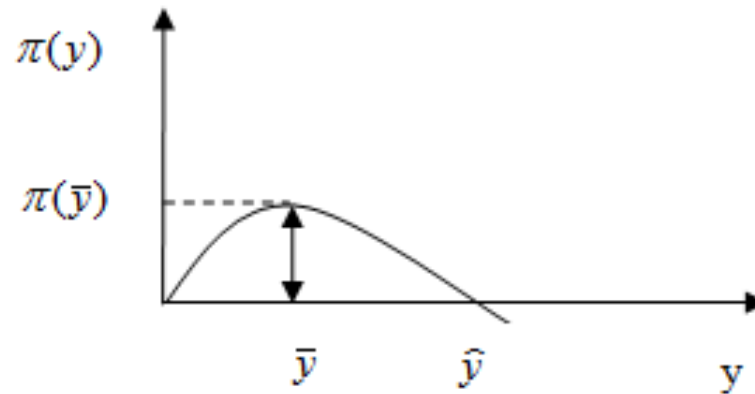
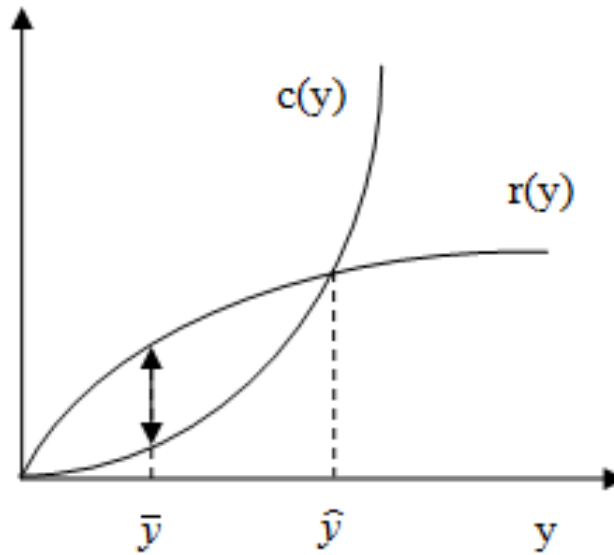
where  $r(y) = p(y) \cdot y$  is the revenue function,  
 $c(y)$  is the cost function.

The first-order condition for the profit maximization problem is

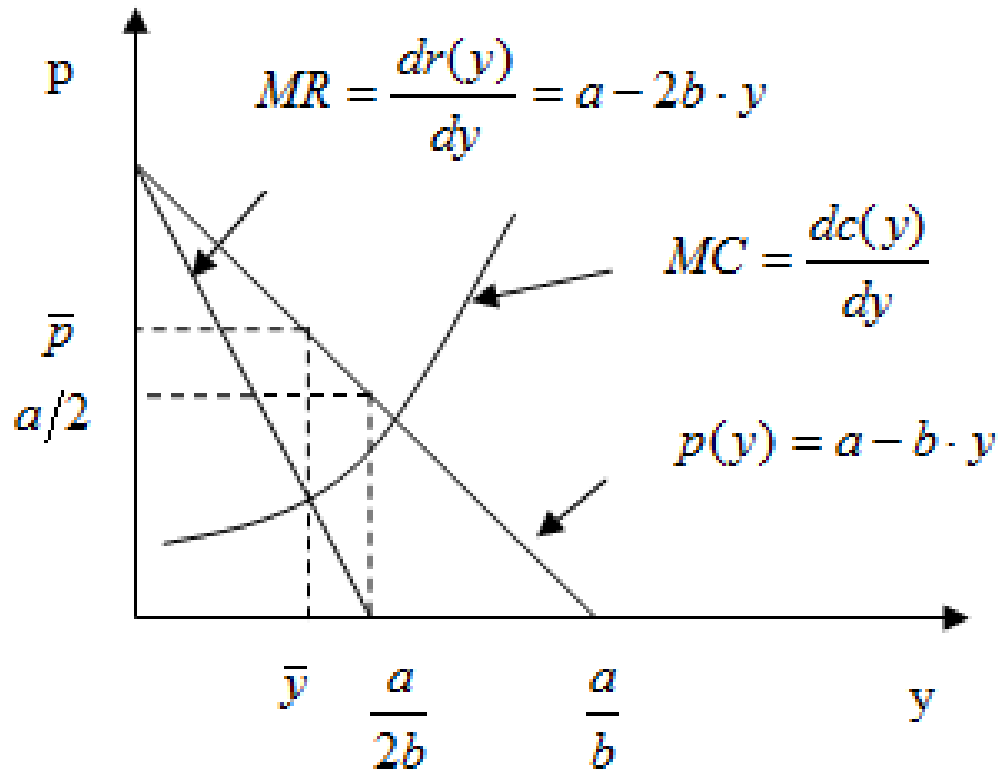
$$\frac{d\pi(y)}{dy} = 0 \quad \Rightarrow \quad \frac{dr(y)}{dy} = \frac{dc(y)}{dy}$$
$$\Rightarrow \quad p(y) \left[ 1 - \frac{1}{|\varepsilon(y)|} \right] = \frac{dc(y)}{dy}.$$

where  $\varepsilon(y) = \frac{dy}{dp} \frac{p}{y}$  - the price elasticity of demand facing the monopolist.

# Monopoly with a Nonlinear Demand Function



# Monopoly with a Linear Demand Function



$$\varepsilon(y) = -\frac{p}{a-p} \quad \Rightarrow \quad |\varepsilon(y)| = 1 \quad \Rightarrow \quad y = \frac{a}{2b}, \quad p = \frac{a}{2}$$

## Example

Let assume that  $y = x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}$ ,  $p(y) = 6y^{-\frac{1}{2}}$ ,  $v_1, v_2$

$$\max_{x_1, x_2} \pi(x_1, x_2) = 6x_1^{\frac{1}{6}} x_2^{\frac{1}{6}} - (v_1 x_1 + v_2 x_2)$$

- $\bar{x}_1 = v_1^{-\frac{5}{4}} v_2^{-\frac{1}{4}}$ ,  $\bar{x}_2 = v_1^{-\frac{1}{4}} v_2^{-\frac{5}{4}}$
- $\bar{y} = v_1^{-\frac{1}{2}} v_2^{-\frac{1}{2}}$

$$\min_{x_1, x_2} v_1 x_1 + v_2 x_2$$

such that

$$y = x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}$$

- $x_1^* = \left( \frac{v_2}{v_1} \right)^{\frac{1}{2}} y^{\frac{3}{2}}$        $x_2^* = \left( \frac{v_1}{v_2} \right)^{\frac{1}{2}} y^{\frac{3}{2}}$
- $c(y) = 2 \cdot v_1^{\frac{1}{2}} \cdot v_2^{\frac{1}{2}} \cdot y^{\frac{3}{2}}$

$$\max_y \pi(y) = 6y^{\frac{1}{2}} - 2 \cdot v_1^{\frac{1}{2}} \cdot v_2^{\frac{1}{2}} \cdot y^{\frac{3}{2}}$$

- $\bar{y} = v_1^{-\frac{1}{2}} v_2^{-\frac{1}{2}}$

# Inefficiency of Monopoly

