Mathematical Economics

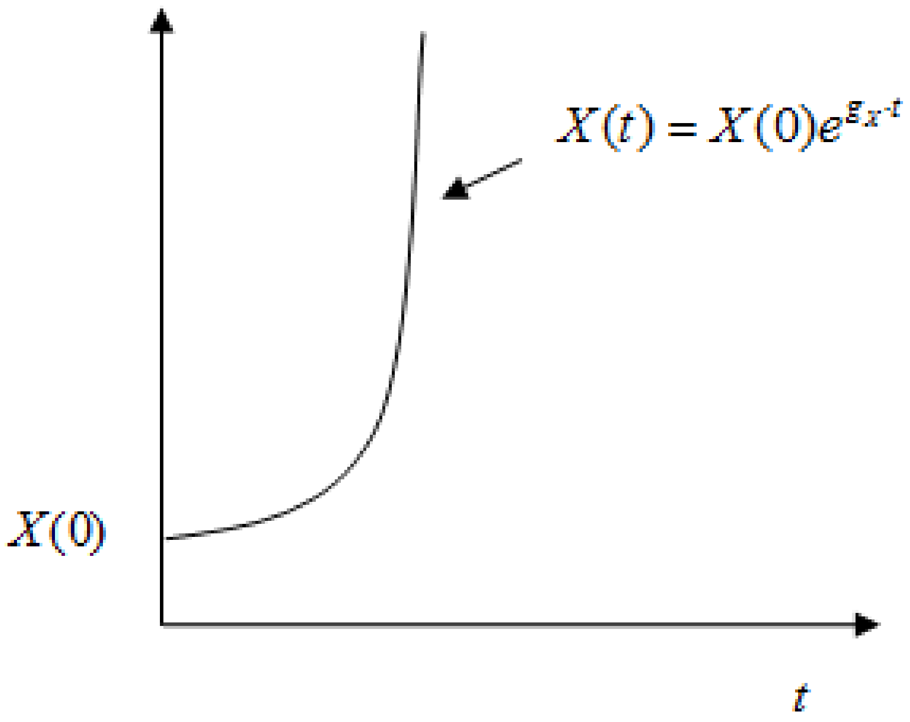
dr Wioletta Nowak

Lecture 7

- Growth rate
- The Solow growth model

Growth Rate

$$g_X = \frac{\dot{X}(t)}{X(t)} \equiv \frac{d \ln X(t)}{dt} \quad \text{where} \quad \dot{X}(t) \equiv \frac{dX(t)}{dt}$$



Properties of Growth Rate

Let $X(t)$, $Y(t)$, then

1. $g_{X\cdot Y} = g_X + g_Y$,
2. $g_{X/Y} = g_X - g_Y$,
3. $g_{X+Y} = \dfrac{X}{X+Y} g_X + \dfrac{Y}{X+Y} g_Y$,
4. $g_{X-Y} = \dfrac{X}{X-Y} g_X - \dfrac{Y}{X-Y} g_Y$,

where g_X is the growth rate of $X(t)$,

g_Y is the growth rate of $Y(t)$.

GDP

- https://data.worldbank.org/indicator

- Exports of goods and services (constant 2010 US$)
- Imports of goods and services (constant 2010 US$)
- Gross capital formation (constant 2010 US$)
- General government final consumption expenditure (constant 2010 US$)
- Households and NPISHs Final consumption expenditure (constant 2010 US$)

- Households and NPISHs Final consumption expenditure (constant 2010 US$) modified

- GDP (constant 2010 US$)
- GDP growth (annual %)

NPISHs - nonprofit institutions serving households

Argentina	2001	2002	2003	2004	2005
Exports of goods and services (constant 2010 US$)	50640355142	5,2207E+10	5,5337E+10	5,9827E+10	6,7536E+10
Imports of goods and services (constant 2010 US$)	36623502524	1,829E+10	2,5161E+10	3,5256E+10	4,0843E+10
Gross capital formation (constant 2010 US$)	39572801787	2,5149E+10	3,4748E+10	4,671E+10	5,3622E+10
General government final consumption expenditure (co	45178652388	4,2884E+10	4,3511E+10	4,4694E+10	4,9098E+10
Households and NPISHs Final consumption expenditure	1,91092E+11	1,5633E+11	1,7267E+11	1,9051E+11	2,042E+11
Final consumption	2,36271E+11	1,9922E+11	2,1618E+11	2,3521E+11	2,533E+11
GDP calculated	2,89861E+11	2,5828E+11	2,8111E+11	3,0649E+11	3,3362E+11
			Share in GDP		
Exports of goods and services (constant 2010 US$)	0,175	0,202	0,197	0,195	0,202
Imports of goods and services (constant 2010 US$)	0,126	0,071	0,090	0,115	0,122
Gross capital formation (constant 2010 US$)	0,137	0,097	0,124	0,152	0,161
General government final consumption expenditure (co	0,156	0,166	0,155	0,146	0,147
Households and NPISHs Final consumption expenditure	0,659	0,605	0,614	0,622	0,612
Final consumption	0,815	0,771	0,769	0,767	0,759
	1,000	1,000	1,000	1,000	1,000

			growth rate		
	2001	2002	2003	2004	2005
Exports of goods and services (constant 2010 US$)		3,1%	6,0%	8,1%	12,9%
Imports of goods and services (constant 2010 US$)		-50,1%	37,6%	40,1%	15,8%
Gross capital formation (constant 2010 US$)		-36,4%	38,2%	34,4%	14,8%
General government final consumption expenditure (constant 2010 US$)		-5,1%	1,5%	2,7%	9,9%
Households and NPISHs Final consumption expenditure (constant 2010 US		-18,2%	10,5%	10,3%	7,2%
Final consumption		-15,7%	8,5%	8,8%	7,7%
GDP calculated		-10,9%	8,8%	9,0%	8,9%
GDP growth (annual %)	-4,4	-10,9	8,8	9,0	8,9

		decomposition of GDP			
Exports of goods and services (constant 2010 US$)		0,5%	1,2%	1,6%	2,5%
Imports of goods and services (constant 2010 US$)		-6,3%	2,7%	3,6%	1,8%
Gross capital formation (constant 2010 US$)		-5,0%	3,7%	4,3%	2,3%
General government final consumption expenditure (constant 2010 US$)		-0,8%	0,2%	0,4%	1,4%
Households and NPISHs Final consumption expenditure (constant 2010 US		-12,0%	6,3%	6,3%	4,5%
Final consumption		-12,8%	6,6%	6,8%	5,9%
GDP		-10,9%	8,8%	9,0%	8,9%
		0,0%	0,0%	0,0%	0,0%
		2002	**2003**	**2004**	**2005**

The Solow Growth Model

$$Y(t) = F(K(t), A(t), L(t)),$$

$Y(t)$ corresponds to GDP,

$K(t)$ is a capital, $L(t)$ is a labour force,

$A(t)$ means technical change.

$Y(t) = F(K(t), A(t)L(t))$ - labour augmenting technology progress (Harrod neutral).

$Y(t) = F(A(t)K(t), L(t))$ - technical change is Solow neutral (capital augmenting technical change)

$Y(t) = A(t)F(K(t), L(t))$ - technical change is Hicks neutral

Properties of Neoclassical Production Function

1. $F: \Re_+^2 \to \Re_+$ is twice differentiable on $\text{int}\,\Re_+^2$.

2. $F(K,0) = F(0,AL) = 0$, $\forall\ K, AL \in \Re_+$. $F(0,0) = 0$

3. $\forall \lambda > 0$, $\forall (K,AL) \in \Re_+^2$ $\quad F(\lambda K, \lambda AL) = \lambda F(K,AL)$ - constant returns to scale.

4. $\forall\ (K,AL),(K',AL') \in \Re_+^2 \quad (K,AL) > (K',AL') \Rightarrow F(K,AL) > F(K',AL')$

   - an increasing function

5. A concave function: $\forall \lambda \in (0,1)$, $\forall\ (K,AL),(K',AL') \in \Re_+^2$

$$F(\lambda K + (1-\lambda)K', \lambda AL + (1-\lambda)AL') \geq \lambda F(K,AL) + (1-\lambda)F(K',AL')$$

6. Inada conditions: $\displaystyle \lim_{K\to 0} \frac{\partial F(K,AL)}{\partial K} = \lim_{(AL)\to 0} \frac{\partial F(K,AL)}{\partial (AL)} = +\infty$

$$\lim_{K\to +\infty} \frac{\partial F(K,AL)}{\partial K} = \lim_{(AL)\to +\infty} \frac{\partial F(K,AL)}{\partial (AL)} = 0.$$

The Solow Model with Labour-augmenting Technological Progress

- Production function is: $Y = K^{\alpha}(AL)^{1-\alpha}$

  $\alpha \in (0, 1)$ is capital share of income. Constant returns to scale.

- Aggregate capital accumulation: $\dot{K} = sY - \delta K$,

  where s is an exogenous saving rate (investment rate),

  δ is a depreciation rate.

- $\frac{\dot{A}}{A} = g$ is an exogenous rate of technology progress ($A(t) = A(0)e^{gt}$).

  Efficiency of labour grows at a constant rate.

- Labour force grows at a constant rate $\frac{\dot{L}}{L} = n$

  (n is a constant growth rate of labour force). $L(t) = L(0)e^{nt}$

If $y = \frac{Y}{A \cdot L}$ is product per efficient unit of labour,

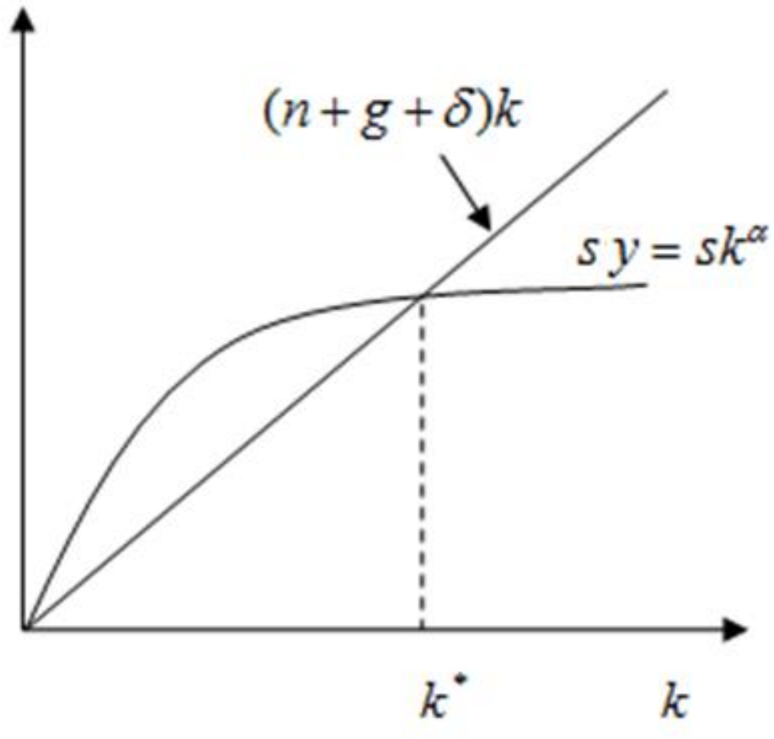
$k = \frac{K}{A \cdot L}$ is capital per efficient unit of labour then

$$y = k^{\alpha}$$

$$\dot{k} = sy - (n + g + \delta)k$$

Steady State Equilibrium

The steady state is at point where $\dot{k} = 0$.



$$k^{*} = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$$

In the steady state growth rates are the following:

$$g_K = n + g,\ g_Y = n + g,\ g_{K/L} = g,\ g_{Y/L} = g$$

An economy approaches to steady state

$$k(t) \to \left(\frac{s}{n+g+\delta} \right)^{\frac{1}{1-\alpha}} = k^*$$

Note:

Solving the equation $\frac{dk}{dt} = sk^{\alpha} - (n+g+\delta)k$ we obtain

$$k(t)^{(1-\alpha)} = \frac{s}{n+g+\delta} + \left(k(0)^{(1-\alpha)} - \frac{s}{n+g+\delta} \right) e^{-(1-\alpha)(n+g+\delta)t}$$

The rate of convergence is $\lambda = (1-\alpha)(n+g+\delta)$

Note: $\dot{k} = \left.\frac{d\dot{k}}{dk}\right|_{k=k^*} (k-k^*) = -(1-\alpha)(n+g+\delta)(k-k^*)$,

$k(t) - k^* = (k(0) - k^*)e^{-\lambda \cdot t}$, where $\lambda = (1-\alpha)(n+g+\delta)$.