Mathematical Economics

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Introduction to Dynamic Optimization

The Ramsey-Cass-Koopmans Growth Model

Introduction to Dynamic Optimization

A typical dynamic optimization problem takes the following form:

$$\max_{c(t)} \quad V = \int_0^T f(k(t), c(t), t)dt$$

subject to

$$\dot{k}(t) = g(k(t), c(t), t) \quad \text{(the equation of motion)}$$
$$k(0) > 0$$

where
- $(0, T)$ is horizon over which the problem is considered.
- V is the value of the objective function as seen from initial moment $t_0 = 0$.
- $c(t)$ is the control variable and objective function V is maximized with respect to this variable.
- $k(t)$ is the state variable and the first constraint describes the evolution of the state variable over time.

In order to solve the optimization program we need first to find the first-order conditions

- We construct the Hamiltonian function

$$H = f\big(k(t), c(t), t\big) + v(t) \cdot g\big(k(t), c(t), t\big)$$

  where $v(t)$ is a Lagrange multiplier

- We take the derivative of the Hamiltonian with respect to the control variable and set it to 0

$$\frac{\partial H}{\partial c} = \frac{\partial f}{\partial c} + \nu \cdot \frac{\partial g}{\partial c}$$

- We take the derivative of the Hamiltonian with respect to the state variable and set it to equal the negative of the derivative of the Lagrange multiplier with respect to time

$$\frac{\partial H}{\partial k} = \frac{\partial f}{\partial k} + \nu \cdot \frac{\partial g}{\partial k} = -\dot{\nu}$$

Transversality condition

- Infinite horizons with *f* of the form

$$f\big(k(t),c(t),t\big) = e^{-\rho \cdot t} \cdot u\big(k(t),c(t)\big)$$

$$\lim_{t \to \infty}\big(k(t) \cdot v(t)\big) = 0$$

- It may happen that the dynamic optimization problem contains more than one control variable and more than one state variable. In that case we need an equation of motion for each state variable. To write the first-order conditions, the algorithm specified above should be modified in the following way:

- The Hamiltonian includes the right-hand side of each equation of motion times the corresponding multiplier.

- We take the derivative of the Hamiltonian with respect to each control variable and set it to 0.

- We take the derivative of the Hamiltonian with respect to each state variable and set it to equal the negative of the derivative of the Lagrange multiplier with respect to time

In economic problems, the objective function is usually of the form

$$f(k(t),c(t),t) = e^{-\rho t}u(k(t),c(t)),$$

where ρ is a constant discount rate and $e^{-\rho t}$ is a discount factor.

$$\max_{c(t)} \quad V = \int_0^T e^{-\rho t}u(k(t),c(t))dt$$

subject to

$$\dot{k}(t) = g(k(t),c(t),t),$$
$$k(0) > 0$$

Present-Value Hamiltonian

$$H = e^{-\rho t} u(k(t), c(t)) + \nu(t) g(k(t), c(t), t)$$

$$\frac{\partial H}{\partial c} = 0$$

$$\frac{\partial H}{\partial k} = -\dot{\nu}$$

$$\frac{\partial H}{\partial \nu} = \dot{k}$$

$$\lim_{t \to \infty} (k(t)\nu(t)) = 0$$

Current-Value Hamiltonian

$$H_c = u(k(t), c(t)) + \nu(t)\, g(k(t), c(t), t)$$

$$\frac{\partial H_c}{\partial c} = 0$$

$$\frac{\partial H_c}{\partial k} = -\dot{\nu} + \rho \cdot \nu$$

$$\frac{\partial H_c}{\partial \nu} = \dot{k}$$

$$\lim_{t \to \infty} e^{-\rho t}\,(k(t)\nu(t)) = 0$$

The Ramsey-Cass-Koopmans Growth Model

The instantaneous utility function of the constant-intertemporal-elasticity-of-substitution (CIES)

$$\int_0^\infty e^{(n-\rho)t}\left(\frac{c(t)^{1-\gamma}-1}{1-\gamma}\right)dt$$

where ρ denotes the constant time preference rate (is the rate at which households discount future utility),

$\gamma > 0$ a constant preference parameter

$(1/\gamma$ - the elasticity of intertemporal substitution),

$c(t) = C(t)/L(t)$ is consumption per capita at time t,

$n = \dfrac{\dot{L}}{L}$ – an exogenous growth rate of population (labour force)

Properties of CIES Utility Function

1. $\varepsilon = -\dfrac{d\ln(c_1/c_2)}{d\ln\left(\dfrac{\partial u/\partial c_1}{\partial u/\partial c_2}\right)} = \dfrac{1}{\gamma}$,

Note: $u(t) = \dfrac{c_1(t)^{1-\gamma}-1}{1-\gamma} + \dfrac{1}{1+\rho}\,\dfrac{c_2(t)^{1-\gamma}-1}{1-\gamma}$,

$\partial u/\partial c_1 = c_1^{-\gamma}$, $\qquad \partial u/\partial c_2 = \dfrac{1}{1+\rho}\,c_2^{-\gamma}$

$\dfrac{\partial u/\partial c_1}{\partial u/\partial c_2} = (1+\rho)\left(\dfrac{c_1}{c_2}\right)^{-\gamma} \qquad \Rightarrow \qquad \left(\dfrac{c_1}{c_2}\right)^{-\gamma} = \dfrac{1}{1+\rho}\,\dfrac{\partial u/\partial c_1}{\partial u/\partial c_2}$

$\ln\left(\dfrac{c_1}{c_2}\right) = -\dfrac{1}{\gamma}\ln\left(\dfrac{1}{1+\rho}\right) - \dfrac{1}{\gamma}\ln\left(\dfrac{\partial u/\partial c_1}{\partial u/\partial c_2}\right)$

Properties of CIES Utility Function

2. Constant risk aversion function $\quad -c \dfrac{d^2u/dc^2}{du/dc} = \gamma$

3. $\displaystyle\lim_{\gamma\to 1}\dfrac{c^{1-\gamma}-1}{1-\gamma} = \lim_{\gamma\to 1}\dfrac{\dfrac{d}{d\gamma}\left(c^{1-\gamma}-1\right)}{\dfrac{d}{d\gamma}(1-\gamma)} = \lim_{\gamma\to 1}\dfrac{-c^{1-\gamma}\ln c}{-1} = \ln c$

Each household chooses, at each moment t,
the level of consumption that maximizes the present value of lifetime utility

$$\int_0^\infty e^{(n-\rho)t} \left(\frac{c(t)^{1-\gamma} - 1}{1-\gamma} \right) dt$$

subject to the dynamic budget constraint

$$\dot{k}(t) = k(t)^\alpha - c(t)e^{-gt} - (n+g+\delta)k(t),$$

and the initial level of capital $k(0) > 0$

where $k = \dfrac{K}{A \cdot L}$ is capital per efficient unit of labour,

$\alpha \in (0, 1)$ is capital share of income,

$\dfrac{\dot{A}}{A} = g$ is an exogenous rate of technology progress,

δ is a depreciation rate.

Note: $Y = K^\alpha (AL)^{1-\alpha}$, $\dot{K} = Y - C - \delta K$

Present-Value Hamiltonian

$$H = e^{(n-\rho)t} \frac{c^{1-\gamma} - 1}{1 - \gamma}$$

$$+ v\left(k^{\alpha} - c \cdot e^{-gt} - (n + g + \delta)k\right)$$

$$\frac{\partial H}{\partial c} = e^{(n-\rho)t} c^{-\gamma} - v e^{-gt} = 0$$

$$\frac{\partial H}{\partial k} = v(\alpha k^{\alpha-1} - (n + g + \delta)) = -\dot{v}$$

$$\frac{\partial H}{\partial v} = k^{\alpha} - c e^{-gt} - (n + g + \delta)k = \dot{k}(t)$$

$$\lim_{t \to \infty} (k(t)v(t)) = 0$$

Current-Value Hamiltonian

$$H_c = \frac{c^{1-\gamma} - 1}{1 - \gamma}$$
$$+ v\left(k^{\alpha} - c \cdot e^{-gt} - (n + g + \delta)k\right)$$

$$\frac{\partial H_c}{\partial c} = c^{-\gamma} - ve^{-gt} = 0$$

$$\frac{\partial H_c}{\partial k} = v(\alpha k^{\alpha-1} - (n + g + \delta)) = -\dot{v} + (\rho - n)v$$

$$\frac{\partial H_c}{\partial v} = k^{\alpha} - ce^{-gt} - (n + g + \delta)k = \dot{k}(t)$$

$$\lim_{t \to \infty} e^{(n-\rho)t}(k(t)v(t)) = 0$$

$$g_c \equiv \frac{\dot{c}(t)}{c(t)} = \frac{1}{\gamma}\left(\alpha k^{\alpha-1} - \delta - \rho\right),$$

$$g_c = g_{Y/L} = g_{K/L} = g$$

Note: $k^* = \left(\dfrac{\alpha}{\delta + \rho + \gamma \cdot g}\right)^{\frac{1}{1-\alpha}}$

The augmented Solow model with human-capital accumulation: the Mankiw-D.Romer-Weil model

$$Y(t) = K(t)^{\alpha} H(t)^{\beta} (A(t)L(t))^{1-\alpha-\beta},$$

$$\alpha > 0, \quad \beta > 0, \quad \alpha + \beta < 1,$$

$$\dot{K}(t) = s_K Y(t) - \delta_K K(t),$$

$$\dot{H}(t) = s_H Y(t) - \delta_H H(t),$$

$$s_K > 0, \; s_H > 0, \; s_K + s_H < 1, \; \delta_K, \delta_H \in (0,1)$$

$H(t)$ is the stock of human capital,

s_K is a fraction of income invested in physical capital,

s_H is a fraction of income invested in human capital,

δ_K is a depreciation rate of physical capital,

δ_H is a depreciation rate of human capital.

The augmented Solow model with human-capital accumulation: the Mankiw-D.Romer-Weil model

$$y(t) = k(t)^{\alpha} h(t)^{\beta}$$

$$\dot{k}(t) = s_K \, y(t) - \left(n + g + \delta_K\right)k(t)$$

$$\dot{h}(t) = s_H \, y(t) - \left(n + g + \delta_H\right)h(t)$$

$$y(t) = \frac{Y(t)}{A(t)L(t)} , \qquad k(t) = \frac{K(t)}{A(t)L(t)} , \qquad h(t) = \frac{H(t)}{A(t)L(t)}$$