Modern Growth Theories
Lecture 4

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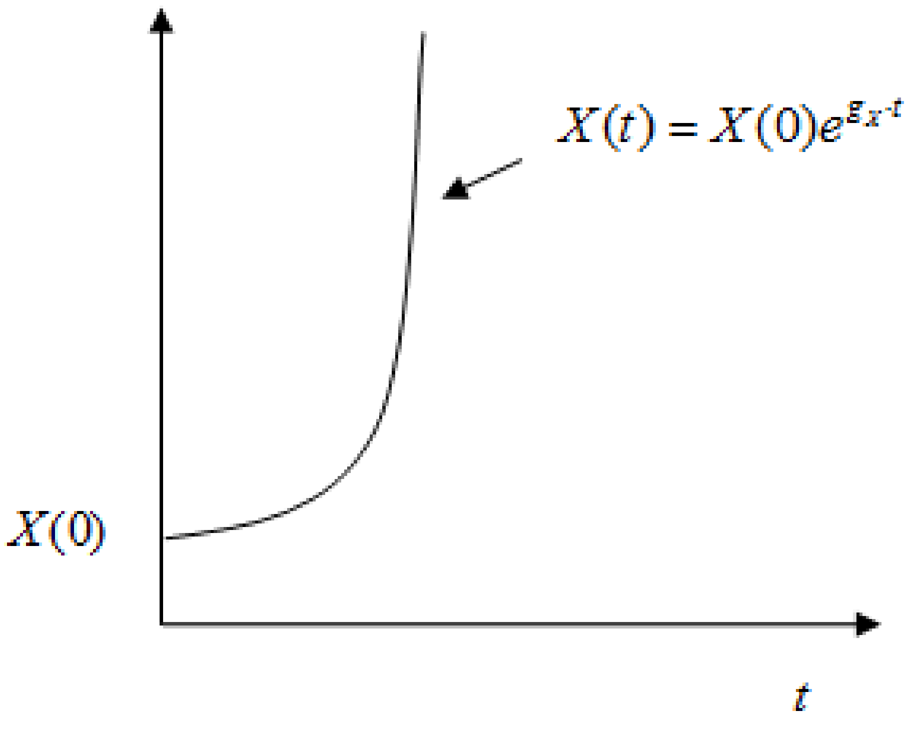
Growth rate

$$g_X = \frac{\dot{X}(t)}{X(t)} \equiv \frac{d \ln X(t)}{dt} \quad \text{where} \quad \dot{X}(t) \equiv \frac{dX(t)}{dt}$$



Properties of growth rate

Let $X(t)$, $Y(t)$, then

1. $g_{X \cdot Y} = g_X + g_Y$,
2. $g_{X/Y} = g_X - g_Y$,
3. $g_{X+Y} = \dfrac{X}{X+Y} g_X + \dfrac{Y}{X+Y} g_Y$,
4. $g_{X-Y} = \dfrac{X}{X-Y} g_X - \dfrac{Y}{X-Y} g_Y$,

where g_X is the growth rate of $X(t)$,

g_Y is the growth rate of $Y(t)$.

The Solow growth model

$$Y(t) = F(K(t), A(t), L(t)),$$

$Y(t)$ corresponds to GDP,

$K(t)$ is a capital, $L(t)$ is a labour force,

$A(t)$ means technical change.

$Y(t) = F(K(t), A(t)L(t))$ - labour augmenting technology progress (Harrod neutral).

$Y(t) = F(A(t)K(t), L(t))$ - technical change is Solow neutral (capital augmenting technical change)

$Y(t) = A(t)F(K(t), L(t))$ - technical change is Hicks neutral

Properties of neoclassical production function

1. $F: \Re_+^2 \to \Re_+$ is twice differentiable on $\operatorname{int} \Re_+^2$.

2. $F(K,0) = F(0,AL) = 0$, $\forall\ K, AL \in \Re_+$. $F(0,0) = 0$

3. $\forall \lambda > 0$, $\forall (K, AL) \in \Re_+^2$ $F(\lambda K, \lambda AL) = \lambda F(K, AL)$ - constant returns to scale.

4. $\forall\ (K, AL), (K', AL') \in \Re_+^2$ $(K, AL) > (K', AL') \Rightarrow F(K, AL) > F(K', AL')$

   - an increasing function

5. A concave function: $\forall \lambda \in (0,1)$, $\forall\ (K, AL), (K', AL') \in \Re_+^2$

$$F(\lambda K + (1-\lambda)K', \lambda AL + (1-\lambda)AL') \geq \lambda F(K, AL) + (1-\lambda)F(K', AL')$$

6. Inada conditions: $\lim\limits_{K \to 0} \dfrac{\partial F(K, AL)}{\partial K} = \lim\limits_{(AL) \to 0} \dfrac{\partial F(K, AL)}{\partial (AL)} = +\infty$

$$\lim_{K \to +\infty} \frac{\partial F(K, AL)}{\partial K} = \lim_{(AL) \to +\infty} \frac{\partial F(K, AL)}{\partial (AL)} = 0 .$$

The Solow model with labour-augmenting technological progress

- Production function is: $Y = K^{\alpha}(AL)^{1-\alpha}$

  $\alpha \in (0, 1)$ is capital share of income. Constant returns to scale.

- Aggregate capital accumulation: $\dot{K} = sY - \delta K$,

  where s is an exogenous saving rate (investment rate).

  δ is a depreciation rate.

- $\dfrac{\dot{A}}{A} = g$ is an exogenous rate of technology progress ($A(t) = A(0)e^{gt}$).

  Efficiency of labour grows at a constant rate.

- Labour force grows at a constant rate $\dfrac{\dot{L}}{L} = n$

  (n is a constant growth rate of labour force). $L(t) = L(0)e^{nt}$

If $y = \dfrac{Y}{A \cdot L}$ is product per efficient unit of labour,

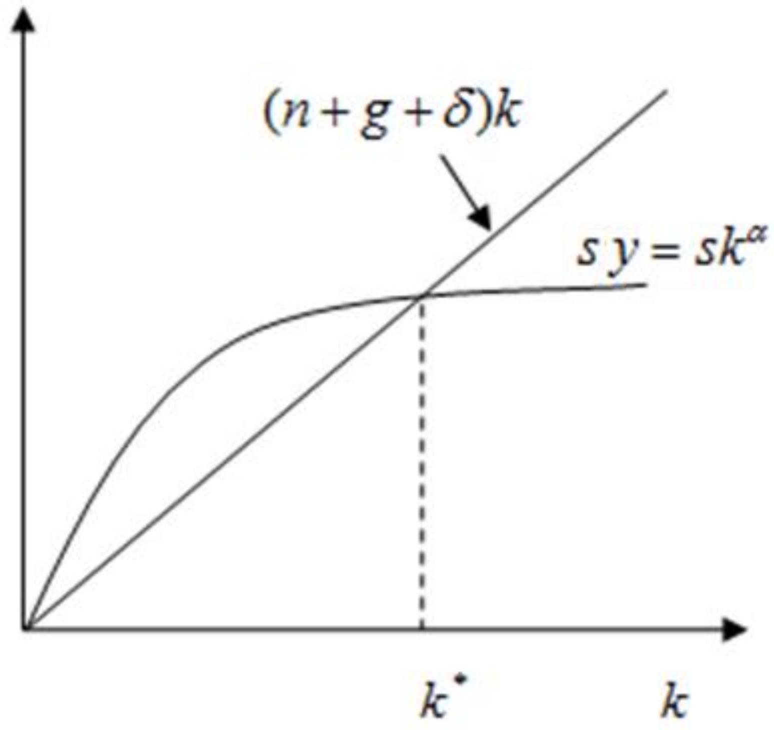
$k = \dfrac{K}{A \cdot L}$ is capital per efficient unit of labour then

$$y = k^{\alpha}$$

$$\dot{k} = sy - (n + g + \delta)k$$

Steady state equilibrium

The steady state is at point where $\dot{k} = 0$.



$$k^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$$

In the steady state growth rates are the following:

$g_K = n + g$, $g_Y = n + g$, $g_{K/L} = g$, $g_{Y/L} = g$

An economy approaches to steady state

$$k(t) \rightarrow \left(\frac{s}{n+g+\delta} \right)^{\frac{1}{1-\alpha}} = k^*$$

Note:

Solving the equation $\frac{dk}{dt} = sk^\alpha - (n+g+\delta)k$ we obtain

$$k(t)^{(1-\alpha)} = \frac{s}{n+g+\delta} + \left(k(0)^{(1-\alpha)} - \frac{s}{n+g+\delta} \right) e^{-(1-\alpha)(n+g+\delta)t}$$

The rate of convergence is $\lambda = (1-\alpha)(n+g+\delta)$

Note: $\dot{k} = \left.\frac{d\dot{k}}{dk}\right|_{k=k^*} (k-k^*) = -(1-\alpha)(n+g+\delta)(k-k^*)$,

$k(t) - k^* = (k(0) - k^*)e^{-\lambda \cdot t}$, where $\lambda = (1-\alpha)(n+g+\delta)$.