Modern Growth Theories
Lectures 5-7

Dr Wioletta Nowak

The Ramsey-Cass-Koopmans growth model

Introduction to dynamic optimization

The extended Solow model with human-capital accumulation

Households

Source: R.J. Barro, X. Sala-i-Martin (2004), *Economic Growth*, MIT Press, p. 86.

- The households provide labour services in exchange for wages, receive interest income on assets, purchase goods for consumption, and save by accumulating assets.
- Identical households - each has the same preference parameters, faces the same wage rate, begins with the same assets per person, and has the same rate of population growth.
- The analysis can use the usual representative-agent framework, in which the equilibrium derives from the choices of a single household.

Households

Source: R.J. Barro, X. Sala-i-Martin (2004), *Economic Growth*, MIT Press, p. 86.

- Each household contains one or more adult, working members of the current generation.
- Adults take account of the welfare and resources of their prospective descendants.
- The intergenerational interaction is modelled by imagining that the current generation maximizes utility and incorporates a budget constraint over an infinite horizon.

The instantaneous utility function of the constant-intertemporal-elasticity-of-substitution (CIES)

$$\int_0^\infty e^{(n-\rho)t}\left(\frac{c(t)^{1-\gamma}-1}{1-\gamma}\right)dt$$

where ρ denotes the constant time preference rate (is the rate at which households discount future utility),

$\gamma > 0$ a constant preference parameter

($1/\gamma$ - the elasticity of intertemporal substitution),

$c(t) = C(t)/L(t)$ is consumption per capita at time t,

$n = \frac{\dot{L}}{L}$ – an exogenous growth rate of population (labour force)

Properties of CIES Utility Function

1. $\varepsilon = -\dfrac{d\ln(c_1/c_2)}{d\ln\left(\dfrac{\partial u/\partial c_1}{\partial u/\partial c_2}\right)} = \dfrac{1}{\gamma}$,

Note: $u(t) = \dfrac{c_1(t)^{1-\gamma}-1}{1-\gamma} + \dfrac{1}{1+\rho}\dfrac{c_2(t)^{1-\gamma}-1}{1-\gamma}$,

$\partial u/\partial c_1 = c_1^{-\gamma}$, $\quad \partial u/\partial c_2 = \dfrac{1}{1+\rho} c_2^{-\gamma}$

$$\frac{\partial u/\partial c_1}{\partial u/\partial c_2} = (1+\rho)\left(\frac{c_1}{c_2}\right)^{-\gamma} \quad \Rightarrow \quad \left(\frac{c_1}{c_2}\right)^{-\gamma} = \frac{1}{1+\rho}\frac{\partial u/\partial c_1}{\partial u/\partial c_2}$$

$$\ln\left(\frac{c_1}{c_2}\right) = -\frac{1}{\gamma}\ln\left(\frac{1}{1+\rho}\right) - \frac{1}{\gamma}\ln\left(\frac{\partial u/\partial c_1}{\partial u/\partial c_2}\right)$$

Properties of CIES Utility Function

2. Constant risk aversion function $-c\dfrac{d^2u/dc^2}{du/dc}=\gamma$

3. $\lim\limits_{\gamma\to 1}\dfrac{c^{1-\gamma}-1}{1-\gamma}=\lim\limits_{\gamma\to 1}\dfrac{\dfrac{d}{d\gamma}\left(c^{1-\gamma}-1\right)}{\dfrac{d}{d\gamma}(1-\gamma)}=\lim\limits_{\gamma\to 1}\dfrac{-c^{1-\gamma}\ln c}{-1}=\ln c$

The Ramsey-Cass-Koopmans Growth Model

Each household chooses, at each moment t,
the level of consumption that maximizes the present value of lifetime utility

$$\int_0^\infty e^{(n-\rho)t}\left(\frac{c(t)^{1-\gamma}-1}{1-\gamma}\right)dt$$

subject to the dynamic budget constraint

$$\dot{k}(t) = k(t)^\alpha - c(t)e^{-gt} - (n+g+\delta)k(t),$$

and the initial level of capital $k(0) > 0$

where $k = \dfrac{K}{A \cdot L}$ is capital per efficient unit of labour,

$\alpha \in (0, 1)$ is capital share of income,

$\dfrac{\dot{A}}{A} = g$ is an exogenous rate of technology progress,

δ is a depreciation rate.

Note: $Y = K^\alpha (AL)^{1-\alpha}$, $\dot{K} = Y - C - \delta K$

Introduction to Dynamic Optimization

A typical dynamic optimization problem takes the following form:

$$\max_{c(t)} \quad V = \int_0^T f(k(t), c(t), t)dt$$

subject to

$$\dot{k}(t) = g(k(t), c(t), t) \quad \text{(the equation of motion)}$$
$$k(0) > 0$$

where

- $(0, T)$ is horizon over which the problem is considered.
- V is the value of the objective function as seen from initial moment $t_0 = 0$.
- $c(t)$ is the control variable and objective function V is maximized with respect to this variable.
- $k(t)$ is the state variable and the first constraint describes the evolution of the state variable over time.

In order to solve the optimization program we need first to find the first-order conditions

- We construct the Hamiltonian function

$$H = f\left(k(t), c(t), t\right) + \nu(t) \cdot g\left(k(t), c(t), t\right)$$

where $\nu(t)$ is a Lagrange multiplier

- We take the derivative of the Hamiltonian with respect to the control variable and set it to 0

$$\frac{\partial H}{\partial c} = \frac{\partial f}{\partial c} + \nu \cdot \frac{\partial g}{\partial c}$$

- We take the derivative of the Hamiltonian with respect to the state variable and set it to equal the negative of the derivative of the Lagrange multiplier with respect to time

$$\frac{\partial H}{\partial k} = \frac{\partial f}{\partial k} + \nu \cdot \frac{\partial g}{\partial k} = -\dot{\nu}$$

Transversality condition

- Infinite horizons with *f* of the form

$$f\left(k(t),c(t),t\right)=e^{-\rho\cdot t}\cdot u\left(k(t),c(t)\right)$$

$$\lim_{t\to\infty}\left(k(t)\cdot v(t)\right)=0$$

- It may happen that the dynamic optimization problem contains more than one control variable and more than one state variable. In that case we need an equation of motion for each state variable. To write the first-order conditions, the algorithm specified above should be modified in the following way:

- The Hamiltonian includes the right-hand side of each equation of motion times the corresponding multiplier.

- We take the derivative of the Hamiltonian with respect to each control variable and set it to 0.

- We take the derivative of the Hamiltonian with respect to each state variable and set it to equal the negative of the derivative of the Lagrange multiplier with respect to time.

In economic problems, the objective function is usually of the form

$$f(k(t), c(t), t) = e^{-\rho t} u(k(t), c(t)) \,,$$

where ρ is a constant discount rate and $e^{-\rho t}$ is a discount factor.

$$\max_{c(t)} \quad V = \int_0^T e^{-\rho t} u(k(t), c(t)) dt$$

subject to

$$\dot{k}(t) = g(k(t), c(t), t) \,,$$

$$k(0) > 0$$

Present-Value Hamiltonian
$H = e^{-\rho t} u(k(t), c(t)) + \nu(t) g(k(t), c(t), t)$
$\frac{\partial H}{\partial c} = 0$
$\frac{\partial H}{\partial k} = -\dot{\nu}$
$\frac{\partial H}{\partial \nu} = \dot{k}$
$\lim_{t \to \infty} (k(t) \nu(t)) = 0$

Current-Value Hamiltonian
$H_c = u(k(t), c(t)) + \nu(t)\, g(k(t), c(t), t)$
$\frac{\partial H_c}{\partial c} = 0$
$\frac{\partial H_c}{\partial k} = -\dot{\nu} + \rho \cdot \nu$
$\frac{\partial H_c}{\partial \nu} = \dot{k}$
$\lim_{t \to \infty} e^{-\rho t}(k(t)\nu(t)) = 0$

The Ramsey-Cass-Koopmans Growth Model

Each household chooses, at each moment t,
the level of consumption that maximizes the present value of lifetime utility

$$\int_0^\infty e^{(n-\rho)t}\left(\frac{c(t)^{1-\gamma}-1}{1-\gamma}\right)dt$$

subject to the dynamic budget constraint

$$\dot{k}(t)=k(t)^{\alpha}-c(t)e^{-gt}-(n+g+\delta)k(t),$$

and the initial level of capital $k(0)>0$

where $k=\dfrac{K}{A\cdot L}$ is capital per efficient unit of labour,

$\alpha\in(0,1)$ is capital share of income,

$\dfrac{\dot{A}}{A}=g$ is an exogenous rate of technology progress,

δ is a depreciation rate.

Note: $Y=K^{\alpha}(AL)^{1-\alpha}$, $\dot{K}=Y-C-\delta K$

Present-Value Hamiltonian
$H = e^{(n-\rho)t}\dfrac{c^{1-\gamma}-1}{1-\gamma} + \nu\left(k^{\alpha} - c\cdot e^{-gt} - (n+g+\delta)k\right)$
$\dfrac{\partial H}{\partial c} = e^{(n-\rho)t}c^{-\gamma} - \nu e^{-gt} = 0$
$\dfrac{\partial H}{\partial k} = \nu(\alpha k^{\alpha-1} - (n+g+\delta)) = -\dot{\nu}$
$\dfrac{\partial H}{\partial \nu} = k^{\alpha} - ce^{-gt} - (n+g+\delta)k = \dot{k}(t)$
$\lim_{t\to\infty}(k(t)\nu(t)) = 0$

Current-Value Hamiltonian
$H_c = \frac{c^{1-\gamma} - 1}{1-\gamma} + v\left(k^{\alpha} - c \cdot e^{-gt} - (n+g+\delta)k\right)$
$\frac{\partial H_c}{\partial c} = c^{-\gamma} - ve^{-gt} = 0$
$\frac{\partial H_c}{\partial k} = v(\alpha k^{\alpha-1} - (n+g+\delta)) = -\dot{v} + (\rho - n)v$
$\frac{\partial H_c}{\partial v} = k^{\alpha} - ce^{-gt} - (n+g+\delta)k = \dot{k}(t)$
$\lim_{t\to\infty} e^{(n-\rho)t}(k(t)v(t)) = 0$

$$g_c \equiv \frac{\dot{c}(t)}{c(t)} = \frac{1}{\gamma}\left(\alpha k^{\alpha-1} - \delta - \rho\right),$$

$$g_c = g_{Y/L} = g_{K/L} = g$$

$$\text{Note: } k^* = \left(\frac{\alpha}{\delta + \rho + \gamma \cdot g}\right)^{\frac{1}{1-\alpha}}$$

The extended Solow model with human-capital accumulation: the Mankiw-D.Romer-Weil model

$$Y(t) = K(t)^{\alpha} H(t)^{\beta} \left(A(t)L(t)\right)^{1-\alpha-\beta},$$

$\alpha > 0$, $\beta > 0$, $\alpha + \beta < 1$,

$$\dot{K}(t) = s_K Y(t) - \delta_K K(t),$$

$$\dot{H}(t) = s_H Y(t) - \delta_H H(t),$$

$s_K > 0$, $s_H > 0$, $s_K + s_H < 1$, $\delta_K, \delta_H \in (0,1)$

$H(t)$ is the stock of human capital,

s_K is a fraction of income invested in physical capital,

s_H is a fraction of income invested in human capital,

δ_K is a depreciation rate of physical capital,

δ_H is a depreciation rate of human capital.

The augmented Solow model with human-capital accumulation: the Mankiw-D.Romer-Weil model

$$y(t) = k(t)^{\alpha} h(t)^{\beta}$$

$$\dot{k}(t) = s_K y(t) - (n + g + \delta_K) k(t)$$

$$\dot{h}(t) = s_H y(t) - (n + g + \delta_H) h(t)$$

$$y(t) = \frac{Y(t)}{A(t)L(t)}, \quad k(t) = \frac{K(t)}{A(t)L(t)}, \quad h(t) = \frac{H(t)}{A(t)L(t)}$$

The augmented Solow model with human-capital accumulation: the Mankiw-D.Romer-Weil model

$$k^* = \left(\frac{s_K^{1-\beta} s_H^{\beta}}{n + g + \delta} \right)^{\frac{1}{1-\alpha-\beta}} \qquad h^* = \left(\frac{s_K^{\alpha} s_H^{1-\alpha}}{n + g + \delta} \right)^{\frac{1}{1-\alpha-\beta}}$$

Human Capital Growth Models

- Two-sector investment-based Uzawa-Lucas model, which focus on the accumulation of human capital (1988)

$$Y(t) = AK(t)^{\alpha}\left(u(t)h(t)L(t)\right)^{(1-\alpha)} h_a^{\psi}(t), \quad 0 < \alpha < 1, \qquad \psi > 0$$

where A is constant technology level,

h is human capital, $(1-\alpha)$ is the share of capital,

h_a is the human capital externality,

ψ is the elasticity of output with respect to the human externality

$$\max_{c,u} \int_0^\infty e^{(n-\rho)t} \left(\frac{c^{1-\gamma} - 1}{1-\gamma} \right) dt$$

subject to

$$\dot{k}(t) = Ak(t)^\alpha \left(u(t)h(t)\right)^{(1-\alpha)} h_a^\psi(t) - c(t) - (n+\delta)k(t),$$

$$\dot{h}(t) = \varphi h(t)(1-u(t))$$

$(1-u(t))$ is time spent for education,

δ is the depreciation rate of physical capital

R&D- based Growth Models

Two-sector scale R&D-based growth models

the Romer model (1990)

Two-sector non-scale R&D-based growth models

the Jones model (1995)

the Eicher-Turnovsky model (1999)

R&D- based Growth Models

- The group of R&D-based growth models consists of the Romer model and its sequential generalizations: the Jones model and the Eicher-Turnovsky model.

- The Romer model predicts scale effects.
- According to the model economic growth is positively correlated to the level of human capital that is often measured by an index of the average years of education or the number of scientists and engineers.

R&D- based Growth Models

- However, the scale effects prediction is not supported by the data.
- In response to the empirical evidence Jones has introduced model where the long run growth is not subject to scale effects.

- Eicher and Turnovsky generalize this model in sense that they make less restrictive assumptions about the production elasticities than Jones.

The Romer model (1990)

$$\dot{A} = \phi A\left((1-\theta)L\right)$$

The Jones Model

$$Y = (A\theta L)^{\alpha} K^{1-\alpha}, \quad \alpha \in (0,1),$$

$$\dot{A} = \phi A^{\varphi}\left((1-\theta)L\right)^{\lambda} - \delta_A A, \quad 0 < \lambda \leq 1, \quad 0 < \theta < 1$$

where A denotes the stock of technology,
θ is a fraction of labour employed in the final good sector,
ϕ - a shift parameter,
φ and λ measure the degree of externalities across time in the R&D process,
δ_A represents the rate of depreciation of technology (the rate of decrease of knowledge).

The Eicher-Turnovsky model

$$Y = \alpha_F A^{\sigma_A} (\theta L)^{\sigma_L} K^{\sigma_K}, \qquad \sigma_A, \sigma_L, \sigma_K \in (0,1)$$

$$\dot{A} = \alpha_J A^{\eta_A} [(1-\theta)L]^{\eta_L} - \delta_A A, \qquad \eta_A, \eta_L \in (0,1)$$

$$\dot{K} = Y - C - \delta_K K$$

where α_F, α_J represent exogenous technological shift factors to the production functions,

$\sigma_A, \sigma_L, \sigma_K$, η_A, η_L are the productive elasticities.

The Determinants of Long-run Growth Rates

Model	The determinants of the long-run growth rate of output (capital) per worker in the steady-state
the Solow model	the rate of exogenous technical change
the Ramsey-Cass-Koopmans model	the rate of exogenous technical change
the Mankiw-D.Romer-Weil model	the rate of exogenous technical change
the Uzawa-Lucas model (without externalities)	the elasticity of output with respect to physical capital, the effectiveness of investment in human capital, the intertemporal elasticity of substitution, the subjective discount rate, the growth rate of labour force,

The Determinants of Long-run Growth Rates

Model	The determinants of the long-run growth rate of output (capital) per worker in the steady-state
the Jones model	the elasticities of labour and knowledge in the R&D sector, the growth rate of labour force,
the Eicher-Turnovsky model	the elasticity of output with respect to capital, knowledge and labour force, the elasticities of labour and knowledge in the R&D sector, the growth rate of labour force,

- In the neoclassical growth models the long-run growth rate depends only on the rate of exogenous technical change.

- In the endogenous models of growth under study the long-run growth rate depends on production parameters and the growth rate of labour force. In the Uzawa-Lucas model preferences additionally influence the growth rate.

Convergence to the Balance Growth Path

- At the heart of current debate on convergence is the question whether growth process in real world represent transitional dynamics or balanced-growth dynamics.

- The answer to the this question is of major significance because the empirical implications as well as the policy implications might differ dramatically depending on whether an economy converges towards its balanced-growth path or grows along it.

Convergence to the Balance Growth Path

- To assess the relative importance of transitional dynamics as opposed to balanced growth dynamics the rate of convergence (the speed at which economy converges towards the long-run equilibrium) should be investigated.

- Transitional dynamics are meaningful if an economy takes a long time to adjust to steady-state. Economies that converge rapidly are close to their steady-state positions.

Convergence to the Balance Growth Path

- Neoclassical and endogenous growth models state convergence to the balanced growth path. However, the models imply different prediction of the qualitative and the quantitative convergence inferences.

- The quantitative convergence implications concern the rate of convergence of the basic per worker variables (output, capital and technology).
- The qualitative convergence implications involve determinants of the rate of convergence and influences of their changes on it.

Convergence to the Balance Growth Path

- In the case of presented models the degree of difficulty of the study of dynamic systems grows successively, from the one to the fourth-order system. A higher-order system provides a much richer dynamics but also complicates computations.

Convergence to the Balance Growth Path

- The dynamics of the Solow model is the simplest one. In the Ramsey-Cass-Koopmans model the linearized dynamics are expressed by the second-order system.
- The dynamics of the Uzawa-Lucas model can be expressed as a third-order system, having a single stable root and one-dimensional stable manifold.
- The most complicate is the analysis of the dynamics of the Eicher-Turnovsky model. In this case the transition path is given by a two-dimensional manifold.

Convergence to the Balance Growth Path

- The conclusions of the Solow model and the Mankiw-D.Romer-Weil model and the Uzawa-Lucas model are that the speed of convergence depends on production parameters and the growth rate of labour force.

- According to the Ramsey-Cass-Koopmans model, the Jones model and the Eicher-Turnovsky model the speed of convergence depends additionally on preference parameters.
- The models suggest different production parameters. The only common production parameters are the elasticity of output with respect to physical capital and the rate of physical capital depreciation

Convergence to the Balance Growth Path

- According to neoclassical models and the investment-based Uzawa-Lucas model output per worker and capital per worker converge to their respective steady-state equilibria at identical and constant rates. In the Uzawa-Lucas model the rates of convergence of output (capital) per worker and human capital are equal.

- The non-scale R&D-based growth models imply that the rates of convergence vary both across time and variables. In the Eicher-Turnovsky model the asymptotic convergence speeds of output (capital) per worker and technology per worker are different.

Model	The rate of convergence of output (capital) per worker (%)	The half-life time of output (capital) per worker (years)
the Solow model	5.1	13.6
the Ramsey-Cass-Koopmans model	10	6.9
the Mankiw-D.Romer-Weil model	2.3	30
the Uzawa-Lucas	16.5	4.2
the Jones model	0.83	83.5
the Eicher-Turnovsky model	1.96	35.4

- The Jones model predicts the slowest rate at which the output per worker converges in the neighbourhood of the steady-state (an annual rate of nearly 1%).
- It takes long time for an out-of-equilibrium economy to adjust to its steady-state.

- The Uzawa-Lucas model predicts the rate of convergence of nearly 17%. The high speed of convergence implies short half-life time in the neighbourhood of the steady-state.
- More precisely, the Jones model predicts that the economy reaches halfway to steady-state in about 84 years, whereas the Uzawa-Lucas model implies that the economy reaches halfway in about 4 years.

- The Mankiw-D.Romer-Weil model and the Eicher-Turnovsky model predict the theoretical value of the rate of convergence consistent with majority of cross-country studies that conclude that income per worker converged at a speed rate of about 2%.

- A small value of the rate of convergence provides strong argument in favour of transitional dynamics as opposed to balanced-growth dynamics.

- The growth process in real world represents rather transitional dynamics than balanced-growth dynamics. Although, the government policy is ineffective with respect to the steady-state growth rates it may influence growth rates along the transition path.

- The transitional dynamics approach seems to be more appropriate in explaining the considerable differences in growth experiences between countries.