Modern Growth Theories
Lecture 9

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Neoclassical Growth Models

The Solow Growth Model (1956)

The Ramsey-Cass-Koopmans Model (1965)

The Mankiw-D.Romer-Weil Model (1992)

The neoclassical model with exogenous savings – the Solow growth model

$$Y = K^{\alpha}(AL)^{1-\alpha}, \qquad \dot{K} = sY - \delta K,$$

$$\frac{\dot{A}}{A} = g, \qquad \frac{\dot{L}}{L} = n$$

$$y = k^{\alpha}$$

$$\dot{k} = sk^{\alpha} - (n + g + \delta)k,$$

where $y = \dfrac{Y}{A \cdot L}$, $k = \dfrac{K}{A \cdot L}$

Y corresponds to GDP,
K is a capital, L is a labour force,
A means technological progress,
$\alpha \in (0, 1)$ is capital share of income,
s is an exogenous saving rate (investment rate),
δ is a depreciation rate,
g is an exogenous rate of technological progress
n is a constant growth rate of labour force

The neoclassical model with endogenous savings – the Ramsey-Cass-Koopmans model

$$U = \int_0^\infty e^{(n-\rho)t} \left(\frac{c^{1-\gamma} - 1}{1-\gamma} \right) dt, \quad Y = K^\alpha (AL)^{1-\alpha},$$

$$\dot{K} = Y - C - \delta K, \quad \frac{\dot{A}}{A} = g, \quad \frac{\dot{L}}{L} = n,$$

$$\max_c \int_0^\infty e^{(n-\rho)t} \left(\frac{c^{1-\gamma} - 1}{1-\gamma} \right) dt$$

subject to

$$\dot{k}(t) = k(t)^\alpha - c(t) e^{-gt} - (n + g + \delta) k(t),$$

$$k(0) > 0$$

$c = C/L$ is consumption per capita,

ρ denotes the constant time preference rate

(is the rate at which households discount future utility),

$\gamma > 0$ is the coefficient of relative risk aversion

(or the inverse of the elasticity of intertemporal substitution

of consumption between two points in time $1/\gamma$)

The extended Solow model with human-capital accumulation: the Mankiw-D.Romer-Weil model

$Y(t) = K(t)^{\alpha} H(t)^{\beta} \left(A(t)L(t)\right)^{1-\alpha-\beta}$,

$\alpha > 0$, $\beta > 0$, $\alpha + \beta < 1$,

$\dot{K}(t) = s_K Y(t) - \delta_K K(t)$,

$\dot{H}(t) = s_H Y(t) - \delta_H H(t)$,

$s_K > 0$, $s_H > 0$, $s_K + s_H < 1$, $\delta_K, \delta_H \in (0,1)$

$H(t)$ is the stock of human capital,

s_K is a fraction of income invested in physical capital,

s_H is a fraction of income invested in human capital,

δ_K is a depreciation rate of physical capital,

δ_H is a depreciation rate of human capital.

The extended Solow model with human-capital accumulation: the Mankiw-D.Romer-Weil model

$$y(t) = k(t)^{\alpha} h(t)^{\beta}$$

$$\dot{k}(t) = s_K y(t) - (n + g + \delta_K)k(t)$$

$$\dot{h}(t) = s_H y(t) - (n + g + \delta_H)h(t)$$

$$y(t) = \frac{Y(t)}{A(t)L(t)}, \quad k(t) = \frac{K(t)}{A(t)L(t)}, \quad h(t) = \frac{H(t)}{A(t)L(t)}$$

The extended Solow model with human-capital accumulation: the Mankiw-D.Romer-Weil model

$$k^* = \left(\frac{s_K^{1-\beta} s_H^{\beta}}{n+g+\delta} \right)^{\frac{1}{1-\alpha-\beta}} \qquad h^* = \left(\frac{s_K^{\alpha} s_H^{1-\alpha}}{n+g+\delta} \right)^{\frac{1}{1-\alpha-\beta}}$$

Human Capital Growth Models

- Two-sector investment-based Uzawa-Lucas model, which focus on the accumulation of human capital (1988)

$$Y(t) = AK(t)^{\alpha} \left(u(t)h(t)L(t)\right)^{(1-\alpha)} h_a^{\psi}(t), \quad 0 < \alpha < 1, \qquad \psi > 0$$

where A is constant technology level,

h is human capital, $(1-\alpha)$ is the share of capital,

h_a is the human capital externality,

ψ is the elasticity of output with respect to the human externality

$$\max_{c,u} \int_0^\infty e^{(n-\rho)t}\left(\frac{c^{1-\gamma}-1}{1-\gamma}\right)dt$$

subject to

$$\dot{k}(t) = Ak(t)^\alpha \left(u(t)h(t)\right)^{(1-\alpha)} h_a^\psi(t) - c(t) - (n+\delta)k(t),$$

$$\dot{h}(t) = \varphi h(t)(1-u(t))$$

$(1-u(t))$ is time spent for education,

δ is the depreciation rate of physical capital

R&D- based Growth Models

Two-sector <u>scale</u> R&D-based growth models

the Romer model (1990)

Two-sector <u>non-scale</u> R&D-based growth models

the Jones model (1995)

the Eicher-Turnovsky model (1999)

R&D- based Growth Models

- The group of R&D-based growth models consists of the Romer model and its sequential generalizations: the Jones model and the Eicher-Turnovsky model.

- The Romer model predicts scale effects.
- According to the model economic growth is positively correlated to the level of human capital that is often measured by an index of the average years of education or the number of scientists and engineers.

R&D- based Growth Models

- However, the scale effects prediction is not supported by the data.
- In response to the empirical evidence Jones has introduced model where the long run growth is not subject to scale effects.

- Eicher and Turnovsky generalize this model in sense that they make less restrictive assumptions about the production elasticities than Jones.

The Romer model (1990)

$$\dot{A} = \phi A((1-\theta)L)$$

The Jones Model

$$Y = (A\theta L)^{\alpha} K^{1-\alpha}\ ,\ \alpha \in (0,1)\ ,$$

$$\dot{A} = \phi A^{\varphi}((1-\theta)L)^{\lambda} - \delta_A A\ ,\quad 0 < \lambda \le 1\ ,\quad 0 < \theta < 1$$

where A denotes the stock of technology,
θ is a fraction of labour employed in the final good sector,
ϕ - a shift parameter,
φ and λ measure the degree of externalities
across time in the R&D process,
δ_A represents the rate of depreciation of technology
(the rate of decrease of knowledge).

The Eicher-Turnovsky model

$$Y = \alpha_F A^{\sigma_A} (\theta L)^{\sigma_L} K^{\sigma_K}, \qquad \sigma_A, \sigma_L, \sigma_K \in (0,1)$$

$$\dot{A} = \alpha_J A^{\eta_A} [(1-\theta)L]^{\eta_L} - \delta_A A, \qquad \eta_A, \eta_L \in (0,1)$$

$$\dot{K} = Y - C - \delta_K K$$

where α_F, α_J represent exogenous technological shift factors to the production functions,

$\sigma_A, \sigma_L, \sigma_K$, η_A, η_L are the productive elasticities.

$$H = \frac{(C/L)^{1-\gamma} - 1}{1-\gamma} + v\left(\alpha_F A^{\sigma_A} (\theta L)^{\sigma_L} K^{\sigma_K} - C - \delta_K K\right) + \mu\left(\alpha_J A^{\eta_A} [(1-\theta)L]^{\eta_L} - \delta_A A\right),$$

$$\frac{\partial H}{\partial C} = 0 \quad \Rightarrow \quad C^{-\gamma} = vL^{1-\gamma},$$

$$\frac{\partial H}{\partial \theta} = 0 \quad \Rightarrow \quad v\sigma_L \frac{Y}{\theta} = \mu\eta_L \frac{J}{1-\theta},$$

$$\frac{\partial H}{\partial K} = -\dot{v} + v\rho \quad \Rightarrow \quad \sigma_K \frac{Y}{K} - \delta_K = \rho - \frac{\dot{v}}{v},$$

$$\frac{\partial H}{\partial A} = -\dot{\mu} + \mu\rho \quad \Rightarrow \quad \frac{v}{\mu}\sigma_A \frac{Y}{A} + \eta_A \frac{J}{A} - \delta_A = \rho - \frac{\dot{\mu}}{\mu},$$

$$\frac{\partial H}{\partial v} = \dot{K} \quad \Rightarrow \quad \dot{K} = Y - C - \delta_K K,$$

$$\frac{\partial H}{\partial \mu} = \dot{A} \quad \Rightarrow \quad \dot{A} = J - \delta_A A,$$

$$\lim_{t\to\infty} vKe^{-\rho t} = \lim_{t\to\infty} \mu A e^{-\rho t} = 0,$$

$$J = \alpha_J A^{\eta_A} [(1-\theta)L]^{\eta_L}.$$

The Determinants of Long-run Growth Rates

Model	The determinants of the long-run growth rate of output (capital) per worker in the steady-state
the Solow model	the rate of exogenous technical change
the Ramsey-Cass-Koopmans model	the rate of exogenous technical change
the Mankiw-D.Romer-Weil model	the rate of exogenous technical change
the Uzawa-Lucas model (without externalities)	the elasticity of output with respect to physical capital, the effectiveness of investment in human capital, the intertemporal elasticity of substitution, the subjective discount rate, the growth rate of labour force,

The Determinants of Long-run Growth Rates

Model	The determinants of the long-run growth rate of output (capital) per worker in the steady-state
the Jones model	the elasticities of labour and knowledge in the R&D sector, the growth rate of labour force,
the Eicher-Turnovsky model	the elasticity of output with respect to capital, knowledge and labour force, the elasticities of labour and knowledge in the R&D sector, the growth rate of labour force,

- In the neoclassical growth models the long-run growth rate depends only on the rate of exogenous technical change.

- In the endogenous models of growth, the long-run growth rate depends on production parameters and the growth rate of labour force. In the Uzawa-Lucas model preferences additionally influence the growth rate.

Convergence to the Balance Growth Path

- At the heart of current debate on convergence is the question whether growth process in real world represent transitional dynamics or balanced-growth dynamics.

- The answer to the this question is of major significance because the empirical implications as well as the policy implications might differ dramatically depending on whether an economy converges towards its balanced-growth path or grows along it.

Convergence to the Balance Growth Path

- To assess the relative importance of transitional dynamics as opposed to balanced growth dynamics the rate of convergence (the speed at which economy converges towards the long-run equilibrium) should be investigated.

- Transitional dynamics are meaningful if an economy takes a long time to adjust to steady-state. Economies that converge rapidly are close to their steady-state positions.

Convergence to the Balance Growth Path

- Neoclassical and endogenous growth models state convergence to the balanced growth path. However, the models imply different prediction of the qualitative and the quantitative convergence inferences.

- The quantitative convergence implications concern the rate of convergence of the basic per worker variables (output, capital and technology).
- The qualitative convergence implications involve determinants of the rate of convergence and influences of their changes on it.

Convergence to the Balance Growth Path

- In the case of presented models the degree of difficulty of the study of dynamic systems grows successively, from the one to the fourth-order system. A higher-order system provides a much richer dynamics but also complicates computations.

Convergence to the Balance Growth Path

- The dynamics of the Solow model is the simplest one. In the Ramsey-Cass-Koopmans model the linearized dynamics are expressed by the second-order system.
- The dynamics of the Uzawa-Lucas model can be expressed as a third-order system, having a single stable root and one-dimensional stable manifold.
- The most complicate is the analysis of the dynamics of the Eicher-Turnovsky model. In this case the transition path is given by a two-dimensional manifold.

Convergence to the Balance Growth Path

- The conclusions of the Solow model and the Mankiw-D.Romer-Weil model and the Uzawa-Lucas model are that the speed of convergence depends on production parameters and the growth rate of labour force.

- According to the Ramsey-Cass-Koopmans model, the Jones model and the Eicher-Turnovsky model the speed of convergence depends additionally on preference parameters.
- The models suggest different production parameters. The only common production parameters are the elasticity of output with respect to physical capital and the rate of physical capital depreciation

Convergence to the Balance Growth Path

- According to neoclassical models and the investment-based Uzawa-Lucas model output per worker and capital per worker converge to their respective steady-state equilibria at identical and constant rates. In the Uzawa-Lucas model the rates of convergence of output (capital) per worker and human capital are equal.

- The non-scale R&D-based growth models imply that the rates of convergence vary both across time and variables. In the Eicher-Turnovsky model the asymptotic convergence speeds of output (capital) per worker and technology per worker are different.

Model	**The rate of convergence of output (capital) per worker (%)**	**The half-life time of output (capital) per worker (years)**
the Solow model	5.1	13.6
the Ramsey-Cass-Koopmans model	10	6.9
the Mankiw-D.Romer-Weil model	2.3	30
the Uzawa-Lucas	16.5	4.2
the Jones model	0.83	83.5
the Eicher-Turnovsky model	1.96	35.4

- The Jones model predicts the slowest rate at which the output per worker converges in the neighbourhood of the steady-state (an annual rate of nearly 1%).
- It takes long time for an out-of-equilibrium economy to adjust to its steady-state.

- The Uzawa-Lucas model predicts the rate of convergence of nearly 17%. The high speed of convergence implies short half-life time in the neighbourhood of the steady-state.
- More precisely, the Jones model predicts that the economy reaches halfway to steady-state in about 84 years, whereas the Uzawa-Lucas model implies that the economy reaches halfway in about 4 years.

- The Mankiw-D.Romer-Weil model and the Eicher-Turnovsky model predict the theoretical value of the rate of convergence consistent with majority of cross-country studies that conclude that income per worker converged at a speed rate of about 2%.

- A small value of the rate of convergence provides strong argument in favour of transitional dynamics as opposed to balanced-growth dynamics.

- The growth process in real world represents rather transitional dynamics than balanced-growth dynamics. Although, the government policy is ineffective with respect to the steady-state growth rates it may influence growth rates along the transition path.

- The transitional dynamics approach seems to be more appropriate in explaining the considerable differences in growth experiences between countries.