MME Mathematical Economics
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Program
Second year, winter semester
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Year (winter/summer semester)

Mathematical Economics
Name of the course

Wioletta Nowak
The Professor

Biography of the Professor

Wioletta Nowak is an assistant professor at the Institute of Economic Sciences at the Faculty of Law, Administration and Economics of the University of Wroclaw, Poland. She has a MSc in theoretical physics and a MA in philosophy from University of Wroclaw, and PhD in economics from the former Higher School of Economics in Wroclaw (Wroclaw University of Economics today). Her research interests span mathematical economics, economic growth, economic development and international economics. She is (co)author of 5 books and over 60 papers.

Requirements for passing a course:

Written exam, five exercises based on exercises from the list below, to pass a student needs 50%, for B – 70% and 90% for A.

Exam exercises:

1. Solve the following utility maximization problem

a) $\max\limits_{x_1,x_2} \left(\frac{1}{2}x_1 + 1\right)(x_2 + 2)$

$2x_1 + x_2 = 8$

b) $\max\limits_{x_1,x_2} x_1 + x_1x_2 + x_2 + 1$

$x_1 + 3x_2 = 9$

c) $\max\limits_{x_1,x_2} x_1^{0,6}x_2^{0,4}$

$3x_1 + 4x_2 = 5$

d) $\max\limits_{x_1,x_2} 3(x_1 + 1)(x_2 + 2)$

$p_1x_1 + p_2x_2 = I$ $\quad p_1, p_2, I > 0$.

2. Find the demanded bundle for a consumer whose utility function and budget constraint are the following

a) $u(x_1, x_2) = (3x_1 + 3)(x_2 + 3)$, $\quad x_1 + 2x_2 = 6$

b) $u(x_1, x_2) = \left(\frac{1}{2}x_1 + 2\right)(x_2 + 4), \quad 2x_1 + x_2 = 10$

c) $u(x_1, x_2) = \left(\frac{1}{3}x_1 + 3\right)(x_2 + 3), \quad 2x_1 + x_2 = 10$

d) $u(x_1, x_2) = (x_1 + 2)\left(\frac{1}{4}x_2 + 2\right), \quad x_1 + 2x_2 = 20$

e) $u(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{4}}, \quad \frac{1}{3}x_1 + 5x_2 = 3$

f) $u(x_1, x_2) = x_1^{\frac{1}{2}} x_2, \quad x_1 + 2x_2 = 6$
g) $u(x_1, x_2) = x_1 + x_2, \quad 2x_1 + x_2 = 8$
h) $u(x_1, x_2) = 4x_1 + x_2, \quad 2x_1 + x_2 = 8$

3. Find the demand functions $x_1(p_1, p_2, I)$ and $x_2(p_2, p_1, I)$ if the budget constraint is $p_1x_1 + p_2x_2 = I$, $p_1, p_2, I > 0$ and the utility function is given by

a) $u(x_1, x_2) = \frac{1}{3}(x_1 + 2)(x_2 + 1),$ b) $u(x_1, x_2) = 3x_1^{\frac{1}{4}} x_2^{\frac{1}{2}},$

c) $u(x_1, x_2) = x_1 x_2^3,$ d) $u(x_1, x_2) = a \ln x_1 + (1 - a) \ln x_2, \ a \in (0,1),$

e) $u(x_1, x_2) = x_1^2 + x_2^2,$ f) $u(x_1, x_2) = \left(x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}\right)^3,$

g) $u(x_1, x_2) = 2\min\{4x_1 + x_2, x_1 + 4x_2\},$ h) $u(x_1, x_2) = \min\{4x_1 + x_2, x_1 + 7x_2\}.$

4. Draw the indifference curve:
a) $u(x_1, x_2) = \frac{1}{2}\min\{x_1, 2x_2\}, \quad u(x_1, x_2) = 4;$
b) $u(x_1, x_2) = \min\{3x_1 + x_2,\ x_1 + 3x_2\}$ passing through point $(4, 4)$;
c) $u(x_1, x_2) = \min\{x_1 + 7x_2, 4x_1 + x_2\}, \quad u(x_1, x_2) = 9.$

5. Solve the expenditure minimization problem

a) $\min_{x_1, x_2} \ p_1x_1 + p_2x_2$, $u = x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}$ b) $\min_{x_1, x_2} \ p_1x_1 + p_2x_2$, $u = x_1^{\frac{1}{4}} x_2^{\frac{1}{2}}$ $p_1, p_2, u > 0.$

6. The utility function is given by

A) $u(x_1, x_2) = x_1^2 + x_2^2;$ B) $u(x_1, x_2) = x_1^{\frac{1}{3}} + x_2^{\frac{1}{3}};$ C) $u(x_1, x_2) = x_1^n + x_2^n;$

D) $u(x_1, x_2) = \frac{1}{2}\ln x_1 + \frac{1}{4}\ln x_2;$ E) $u(x_1, x_2) = x_1^{\frac{1}{n}} + x_2^{\frac{1}{n}}, \quad n \neq 0.$

Check the following relationships

a) $\varphi(p,I) = f(p,v(p,I))$ – the Marshallian demand at income I is the same as the Hicksian demand at utility $v = v(p,I)$

b) $f(p,u) = \varphi(p,e(p,u))$ – the Hicksian demand at utility u is the same as the Marshallian demand at income $e(p,u)$

c) $e(p,v(p,I)) = I$ – the minimum expenditure to reach utility $u = v(p,I)$ is I

d) $v(p,e(p,u)) = u$ – the maximum utility from income $e(p,u)$ is u

e) $\frac{\partial \varphi_i(p,I)}{\partial p_j} = \frac{\partial f_i(p,u)}{\partial p_j} - \frac{\partial \varphi_i(p,I)}{\partial I} \cdot \varphi_j(p,I)$, $\quad i, j = 1,2 \;\; i \neq j$ – the Slutsky equation

7. Check the returns to scale for the following technologies

a) $f(x_1, x_2) = 3x_1 + x_2$, b) $f(x_1, x_2) = \sqrt{x_1 + 2x_2}$, c) $f(x_1, x_2) = x_1^{1/4} x_2^{3/4}$,

d) $f(x_1, x_2) = x_1^2 x_2^3$, e) $f(x_1, x_2) = \left(x_1^{1/4} + x_2^{1/4}\right)^4$, f) $f(x_1, x_2) = \sqrt{x_1} + x_2^2$.

8. For the following technologies

A) $y = A\left(ax_1^{\rho} + (1-a)x_2^{\rho}\right)^{\frac{1}{\rho}}$, B) $y = A\left(ax_1^{-\rho} + (1-a)x_2^{-\rho}\right)^{-\frac{1}{\rho}}$, $\quad 0 \neq \rho > -1$, $\; 0 < a < 1$, $\; A > 0$

C) $y = Ax_1^c x_2^d$, $A, c.d > 0$

compute:

a) the marginal product of capital and marginal product of labour,

b) the technical rate of substitution,

c) the output elasticity of capital and output elasticity of labour,

d) the elasticity of substitution,

e) the elasticity of scale,

f) $\lim_{\rho \to 0} y$.

9. A firm has a production function given by

A) $y = 4x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}$; B) $y = 3x_1^{\frac{1}{4}} x_2^{\frac{1}{2}}$; C) $y = 5x_1^{\frac{1}{3}} x_2^{\frac{1}{2}}$; D) $y = 12x_1^{\frac{1}{6}} x_2^{\frac{1}{3}}$.

a) What are the factor demand functions?

b) What are the conditional factor demand functions?

c) What is the cost function?

d) What is the supply function?

e) What is the profit function?

(Do calculations for long and short run).

10. For the technologies $y = \min\left\{\frac{1}{4}x_1, 2x_2\right\}$ and $y = \frac{x_1}{2} + x_2$ compute:

a) the total cost and the cost of production of 1 unit of output,

b) the marginal and average cost.

Assume that prices of inputs are (4, 6).

11. A monopoly has a production function $y(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{3}}$. Solve the profit maximization problem and cost minimization problem in long and short run if

a) $p(y)=6y^{-\frac{1}{2}}$, $v_1=2$, $v_2=3$,
b) $p=2$, $v_1(x_1)=2x_1$, $v_2(x_2)=3x_2$,
c) $p(y)=6y^{-\frac{1}{2}}$, $v_1(x_1)=2x_1$, $v_2(x_2)=3x_2$.

12. Suppose that we have two firms that face linear demand curve $p=200-\frac{1}{2}(y_1+y_2)$ and their cost functions are $c_1(y_1)=\frac{1}{2}y_1^2$, $c_2(y_2)=10y_2$, respectively.
a) Compute the Cournot equilibrium amount of output for each firm and firms' profits.
b) If firm 2 behaves as a follower and firm 1 behaves as a leader, compute the Stackelberg equilibrium amount of output for each firm and firms' profits.
Please repeat calculations if:
A) $p=40-\frac{1}{2}(y_1+y_2)$; $c_1(y_1)=2y_1^2$; $c_2(y_2)=\frac{1}{2}y_2^2$;
B) $p=30-3(y_1+y_2)$; $c_1(y_1)=3y_1$; $c_2(y_2)=y_2$.
C) $p=100-2(y_1+y_2)$, $c_1(y_1)=2y_1$, $c_2(y_2)=\frac{1}{4}y_2^2$.
D) $p=40-(y_1+y_2)$, $c_1(y_1)=\frac{1}{4}y_1^2$, $c_2(y_2)=3y_2$.

13. The traders' utilities are given by $u^1(x_1,x_2)=x_1x_2^2$ and $u^2(x_1,x_2)=x_1^{1/2}x_2^{1/2}$. Their initial endowments are the following $a^1=(2,2)$ and $a^2=(4,4)$. Traders come to a market and exchange commodities to maximize their utilities. Compute the price vector in equilibrium. Compare the utilities before and after the exchange.
Please repeat calculations if:
A) $a^1=(1,4)$, $a^2=(2,1)$, $u^1(x_1,x_2)=x_1^3x_2$, $u^2(x_1,x_2)=x_1^{\frac{3}{2}}x_2^{\frac{3}{4}}$.
B) $a^1=(10,10)$, $a^2=(20,5)$, $u^1(x_1,x_2)=x_1^{2/3}x_2^{1/3}$, $u^2(x_1,x_2)=x_1^{1/3}x_2^{1/2}$.
C) $a^1=(3,9)$, $a^2=(1,3)$, $u^1(x_1,x_2)=x_1x_2^{1/3}$, $u^2(x_1,x_2)=x_1^3x_2$.

Sources:

1. Lecture notes https://prawo.uni.wroc.pl/node/26168